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Model-matching methods and distributed control of networks consisting of a class of heterogeneous dynamic agents

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ABSTRACT
Many recent results on distributed control of multi-agent networks rely on a number of simplifying assumptions that facilitate the solution of regulation problems associated with large-scale networked systems. Identical subsystem models are a typical assumption made in networked control systems which often fail in practice. In this paper, we propose a systematic method for removing this assumption, leading to a general approach to distributed-control design for stabilising networks of multiple non-identical dynamic agents. Local subsystems represented as autonomous dynamic agents are assumed to share a set of structural properties, (controllability) indices. Our approach relies on the solution of certain model-matching type problems using local state-feedback and state/input-matrix transformations that map local dynamics to a target system, selected to minimise joint control effort. By adapting well-established distributed LQR control design methodologies to our framework, the stabilisation problem of networks of non-identical dynamic agents is solved. The applicability of our approach is illustrated via a synchronisation example of eleven harmonic oscillators with non-identical dynamics communicating over a connected graph.

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Heterogeneous dynamics; model-matching; controllability indices; distributed control; distributed LQR

1. Introduction

Multi-agent networked systems composed of several interacting subsystems and multiple control units have attracted considerable interest in recent years due to their association with a broad spectrum of applications (García et al., 2020; Manfredi & Tucci, 2017; Vlahakis et al., 2019). For reasons related to complexity, robustness, reliability and effectiveness of communications, distributed control has emerged as the dominant design methodology in this domain (Esfahani & Khorasani, 2016; Franzè et al., 2018; Scattolini, 2009; Vlahakis et al., 2018).

Distributed control is a challenging research field, dealing with the problem of how a global decision making task can be shared across several *local* controllers. The concept of decentralisation was initially introduced in socio-economic literature to address the problem of efficient capital allocation (Camacho, 1970; McFadden, 1969). Thereafter, the term *distributed control* has been widely adopted by the control community in problems where centralised schemes result in prohibitive data handling and computational requirements.

Decentralised and distributed control structures, whether physically imposed or consciously conceived, often give rise to elaborate control design problems which are further amplified by the presence of communication links in the overall system. In traditional applications, ideal communication channels are often assumed. In practice, however, information exchange carried out over communication channels is limited by communication delays, limited bandwidth and cross-talk. Overall, distributed control lies in the intersection of control and communication theories (Hespanha et al., 2007) and represents a substantial challenge for the efficient control design of networked systems in real applications.

Multi-vehicle formation problems, gossip algorithms, distributed estimation in networks, synchronisation of multiple power units in smart power grids are typical examples arising in multi-agent control. Various problems in networked systems often appear as state-agreement, synchronisation and consensus tasks (Chen & Dimarogonas, 2019; de Galland & Hendrickx, 2019). From a practical point of view, network stabilisation is one of the most challenging problems in multi-agent network control (Olfati-Saber, 2006; Olfati-Saber et al., 2007). Many significant results in this direction have made use of systems and control theory along with algebraic tools from graph theory (Mesbahi & Egerstedt, 2010). Frequently, the network stabilisation task is formulated as a structured optimal control problem (Keviczky et al., 2006). Despite the fact that optimal control theory is standard for centralised configurations, distributed optimal control design of large-scale structured systems is a considerable challenge.

Significant results in stability analysis of distributed control schemes have led to a deeper understanding of the relationship between algebraic properties of the interconnection graph and network stability. An insightful contribution in this direction has been the work of Fax and Murray (2004), which establishes stability analysis tools for networks of identical linear dynamical systems over generic graphs. In this work, the authors focus on distributed formation control and develop information

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exchange protocols that guarantee formation stability and performance and are robust to changes in the communication topology of the network.

Several contributions in the field of distributed control are the result of simplifying assumptions, including homogeneous subsystem dynamics, bidirectional communication links, dynamically decoupled subsystems, etc. Based on the spectral properties of the incidence matrix of the communication graph, stability analysis in state-agreement and formation control problems of multi-agent networks with single-integrator dynamics has been investigated in Dimarogonas and Johansson (2010). A thorough procedure for designing distributed controllers for a class of coupled systems based on a decomposition approach has been presented in Massioni and Verhaegen (2009). The method relies on certain structural properties satisfied by the system matrices and the repetitive structure of the overall scheme. Authors optimising a multi-objective function subject to linear matrix inequality constraints derive explicit expressions for computing distributed feedback controllers with H_∞ and H_2 guaranteed performance, respectively. In Hengster-Movric et al. (2015), a distributed static output feedback control protocol for state synchronisation is proposed. In the paper, global optimality conditions are derived with respect to a quadratic performance index under the assumption of identical linear dynamics.

A powerful method for distributed LQR design for stabilising networks of homogeneous dynamically decoupled linear systems is presented in Borrelli and Keviczky (2008). Therein, a distributed regulation task is formulated as a large-scale optimal control problem where the performance index couples the behaviour of the systems. Authors propose a *top-down* approach in which a centralised optimal LQR controller is designed and, then, approximated by a distributed control scheme whose stability is guaranteed by the stability margins of LQR control. A complementary method for designing distributed LQR control is presented in Deshpande et al. (2012). This work proposes a *bottom-up* approach in which optimal interactions between self-stabilising agents are defined to minimise an upper bound on an aggregate LQR criterion. Further, an analysis of the proposed control law in the presence of communications delays is carried out and a bound on the maximum delay accommodated by the proposed controller is established.

In the present paper, we focus on multi-agent networks composed of non-identical, dynamically decoupled systems. Our method extends the results of Borrelli and Keviczky (2008) and Deshpande et al. (2012) on distributed LQR control to the heterogeneous-dynamics case. Specifically, we assume that systems constituting a network have common controllability indices (Antsaklis & Michel, 2006) but, otherwise, different dynamics.

We follow a model-matching approach to solve the network stabilisation problem. Our *model-matching* definition gives considerable flexibility, as the output matrices of the mapped systems are required to be square and invertible but are otherwise arbitrary. Here, all agents match the input-to-state part of a target system via state-feedback control, input matrix scaling transformations and a change of coordinates. The target model can be selected so that a measure of the joint model-matching control effort is minimised. This allows closed-loop network performance to be effectively determined by the tuning of an LQR global optimality criterion which is defined and optimised in the second stage of our approach. This extends the state-feedback distributed control schemes presented in Borrelli and Keviczky (2008) and Deshpande et al. (2012), leading to a solution of the stabilisation problem for networks with non-identical agent dynamics. Preliminary results of this effort have been presented in Vlahakis and Halikias (2018a) and Vlahakis and Halikias (2018b). In Section 4.7, we highlight how our results can be extended in the context of distributed nonlinear control.

The main contributions of this paper and significant challenges visited here are summarised as follows.

(1) We propose a new control algorithm for solving large-scale stabilisation problems over networks of non-identical dynamic agents. The control scheme proposed is obtained via a two-step approach, in the second stage of which an upper bound of a joint LQR criterion is optimised. The performance of the overall control system can be tuned by means of weighting matrices similar to the standard LQR control problem.
(2) We identify a class of agents characterised by identical controllability indices and propose a state-feedback model-matching technique whereby agents are locally mapped to a target system. This effectively permits the application of distributed control, the design of which is considerably simplified due to dynamical match of the underlying agents. By minimising a well-defined cost described as a joint model-matching control effort, we select an optimal target system from the permissible class. This effectively enables the overall control system to be tuned by an aggregate LQR criterion.
(3) Our technique extends two well-established results on distributed LQR control beyond their original scope, specifically, to the case of networks of heterogeneous dynamic agents. Due to the feedback formulation of our model-matching approach, various distributed state-feedback control methods originally established for homogeneous multi-agent systems can be adapted to our design setup.
(4) Our method is sufficiently versatile and can be extended to more intricate settings, such as networks with heterogeneous *nonlinear* agents.
(5) We establish stability conditions which is a major challenge due to the presence of non-identical agents. We tackle this by means of a simple but powerful model-matching technique which facilitates the design of a distributed control scheme consisting of a combination of local and *neighboring* state-feedback control.

The remaining of the paper is organised in seven sections. In Section 2, notation and preliminaries on graph theory are given. Section 3 presents the problem examined in this study, along with its motivation in the context of existing literature. Our main results, namely, the model-matching feedback control scheme and the distributed LQR-based control design are presented in Sections 4 and 5, respectively. These are followed by a numerical example in Section 6. Finally, Section 7 discusses our main results and future research directions.

2. Notation and preliminaries

The field of real and complex numbers is denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. $\mathbb{R}^n$ denotes the n-dimensional vector space over the field $\mathbb{R}$ and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. Let $x_1, \ldots, x_n$ be vectors not necessarily of the same dimensions. Then, $\hat{x} = \mathrm{Col}(x_1, \ldots, x_n) = [x_1' \cdots x_n']'$. Let $a_1, \ldots, a_n \in \mathbb{R}$, then $A = \mathrm{diag}(a_1, \ldots, a_n)$ is a diagonal matrix, $a_1, \ldots, a_n$ being its diagonal entries. Note that if $a_1, \ldots, a_n$ are square matrices (not necessarily of the same dimensions), $A = \mathrm{diag}(a_1, \ldots, a_n)$ is a block-diagonal matrix. We denote by $\det(A)$ the determinant of a square matrix A. The column space of a matrix is the set of all linear combinations of its columns. Let $a_1, \ldots, a_m \in \mathbb{R}^n$ and $A = [a_1 \cdots a_m] \in \mathbb{R}^{n\times m}$, then $\mathrm{Im}(A)$ denotes the column space of A and $\mathrm{Im}(A) = \mathrm{span}(a_1, \ldots, a_m)$. Let $\mathcal{X} \subseteq \mathbb{R}^n$. The dimension of $\mathcal{X}$ is denoted by $\dim(\mathcal{X})$. The transpose of ξ is denoted by ξ'. The identity matrix of dimension $m \times m$ is denoted by $I_m \in \mathbb{R}^{m\times m}$. The $n \times m$ zero matrix is denoted by $0_{n\times m}$ unless the dimensions are obvious in which case (part of) the subscript will be omitted. Matrix $\Xi \in \mathbb{R}^{n\times n}$ is called symmetric if $\Xi' = \Xi$. $\mathbf{Re}(s)$ denotes the real part of $s \in \mathbb{C}$. The set of complex numbers with non-positive real part is denoted by $\mathbb{C}_- = \{s \in \mathbb{C} : \mathbf{Re}(s) \leq 0\}$. Similarly, $\mathbb{C}_{--} = \{s \in \mathbb{C} : \mathbf{Re}(s) < 0\}$. $A \otimes B$ denotes the Kronecker product of matrices A and B. If Ξ is symmetric, $\lambda_i(\Xi)$ denotes the ith eigenvalue of Ξ ordered in non-decreasing order of magnitude and $S(\Xi)$ is the spectrum of Ξ. Matrix $\Xi \in \mathbb{R}^{n\times n}$ is called stable or Hurwitz if all its eigenvalues have negative real part, i.e., $\lambda_i(\Xi) \in \mathbb{C}_{--}$, $i = 1, \ldots, n$. We will make use of the following:

Proposition 2.1 (Borrelli & Keviczky, 2008): *Consider matrices $A_1, A_2 \in \mathbb{R}^{m\times m}$ and $\Xi \in \mathbb{R}^{n\times n}$, and let $\bar{A}_1 = I_n \otimes A_1$ and $\bar{A}_2 = \Xi \otimes A_2$ with $\bar{A}_1, \bar{A}_2 \in \mathbb{R}^{nm\times nm}$. Then, $S(\bar{A}_1 + \bar{A}_2) = \bigcup_{i=1}^{n} S(A_1 + \lambda_i(\Xi)A_2)$, where $\lambda_i(\Xi) \in \mathbb{C}$ represents the ith eigenvalue of Ξ.*

2.1 Graph theory preliminaries - undirected graphs

An undirected graph $\mathcal{G}$ is defined as the ordered pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes (or vertices) $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges (i, j) with $i \in \mathcal{V}$, $j \in \mathcal{V}$, $i \neq j$. The orientation of all edges is bidirectional, i.e. if $(i, j) \in \mathcal{E}$, then $(j, i) \in \mathcal{E}$. The degree d_j of a graph vertex j is the number of edges that start from j. Let $d_{\max}(\mathcal{G})$ denote the maximum vertex degree of $\mathcal{G}$. We denote by $\mathcal{A}(\mathcal{G})$ the $(0, 1)$ adjacency matrix of $\mathcal{G}$. In particular, $\mathcal{A}_{ij} = 1$ if $(i, j) \in \mathcal{E}$ with $i, j = 1, \ldots, N$ and $i \neq j$, otherwise $\mathcal{A}_{ij} = 0$. Let $j \in \mathcal{N}_i$ if $(i, j) \in \mathcal{E}$ and $i \neq j$. We call $\mathcal{N}_i$ the neighbourhood of node i. The adjacency matrix $\mathcal{A}(\mathcal{G})$ of undirected graphs is symmetric. We define the Laplacian matrix as $\mathcal{L}(\mathcal{G}) = D(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, where $D(\mathcal{G})$ is the diagonal matrix of vertex degrees d_i (also called the valence matrix). The Laplacian matrix of an undirected graph is a symmetric positive semidefinite matrix. Let $S(\mathcal{L}(\mathcal{G})) = \{\lambda_1(\mathcal{L}(\mathcal{G})), \ldots, \lambda_N(\mathcal{L}(\mathcal{G}))\}$ be the spectrum of the Laplacian matrix $\mathcal{L}$ associated with an undirected graph $\mathcal{G}$ arranged in non-decreasing semi-order, with $\lambda_i(\mathcal{L}(\mathcal{G})) \geq 0$, $i = 1, \ldots, N$. For a survey on spectral graph theory, see Mohar (1991). The following Proposition is derived from Proposition 2.1 in a straightforward manner.

Proposition 2.2: *Let A, B be matrices of appropriate dimensions and $\mathcal{L}$ be the Laplacian matrix of a graph $\mathcal{G}$ over N vertices. Let also $S(\mathcal{L}) = \{\lambda_1(\mathcal{L}), \ldots, \lambda_N(\mathcal{L})\}$. Then, $S(I_N \otimes A + \mathcal{L} \otimes B) = \bigcup_{i=1}^{N} S(A + \lambda_i(\mathcal{L})B)$.*

Definition 2.1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a connected graph over N vertices. Let also matrix $\Xi \in \mathbb{R}^{mN\times nN}$ be partitioned into N^2 blocks of equal dimensions $(m \times n)$, the (i, j)-block being defined as $\Xi_{ij} = \Xi[(i-1)m + 1 : im, (j-1)n + 1 : jn]$, $i, j = 1, \ldots, N$, such that $\Xi_{ij} = 0_{n\times m}$ if $(i, j) \notin \mathcal{E}$. Then,

$$\mathcal{K}^N_{m,n}(\mathcal{G}) = \{\Xi \in \mathbb{R}^{mN\times nN} \mid \Xi_{ij} = 0_{m\times n} \text{ if } (i,j) \notin \mathcal{E}, \text{ with } i,j = 1, \ldots, N \text{ and } i \neq j\} \tag{1}$$

denotes a class of structured matrices.

3. Network setup and problem statement

Consider a network of N dynamically decoupled agents with dynamics described by the following state-space equations:

$$\dot{x}_i = A_i x_i + B_i u_i, \; x_i(0) = x_{0,i}, \; i = 1, \ldots, N, \tag{2}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, while $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ represent state and input vectors, respectively. To accommodate network's description as a dynamic system, we adopt a graph representation whereby agent-i is associated with the ith node of a bidirectional connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$. Collecting now the states of all neighbours of agent-i, we define $\hat{x}_i = \mathrm{Col}(x_i, x_{i_1}, \ldots, x_{i_k})$, where $i_1, \ldots, i_k \in \mathcal{N}_i$. Similarly, collecting state and input variables of all nodes, we define aggregate state and input vectors as $\hat{x} = \mathrm{Col}(x_1, \ldots, x_N)$ and $\hat{u} = \mathrm{Col}(u_1, \ldots, u_N)$, respectively. In the sequel, we refer to (x_i) $\hat{x}_i$ as the (*individual*) *local* state of agent-i, while $\hat{x}$, $\hat{u}$ are referred to as *global* state and input vectors, respectively. Throughout the paper, we assume the following:

Assumption 3.1:

(A1) Individual states x_i, $i = 1, \ldots, N$, are perfectly measured.
(A2) Systems (A_i, B_i), $i = 1, \ldots, N$, are controllable.
(A3) For LQR weighting matrices $Q_1 = Q_1' \geq 0$, $Q_2 = Q_2' \geq 0$ (specified later), let $C_1 \in \mathbb{R}^{n\times n}$, $C_2 \in \mathbb{R}^{n\times n}$ such that $C_1'C_1 = Q_1$, $C_2'C_2 = Q_2$. Then, systems (A_i, C_1), (A_i, C_2), $i = 1, \ldots, N$, are observable.
(A4) Systems (A_i, B_i), $i = 1, \ldots, N$, have identical sets of controllability indices.
(A5) The presence of edge $(i, j) \in \mathcal{E}$ indicates that agent-i transmits its state information to agent-j and vice versa.

Remark 3.1: Assumption 3.1-(A1) is essential for the subsequent analysis and control design presented in this work, since the proposed control scheme relies on state-feedback strategies. Lifting this assumption would require the development of output-feedback techniques (perhaps by means of Kalman Filtering), which are more demanding and challenging especially in a distributed framework. Our ultimate aim here is to guarantee stability of networks consisting of heterogeneous agents based on a distributed LQR control scheme. In Section 5, it will become clear that the large stability margins of LQR control are vital for approximating a centralised controller by a distributed scheme maintaining stability. Besides, it is well known that these stability properties of LQR control vanish, e.g. in the presence of state-estimators, and thus extensions of our method to the case of output feedback control are not straightforward.

Remark 3.2: Assumption 3.1-(A5) emphasises that the graph of the network is undirected. The results of the paper can be extended to the case of directed graphs. This requires a more complex analysis which can be found in Vlahakis (2020, Chapter 4).

Under Assumption 3.1, we wish to design a distributed (LQR-based) state-feedback controller that 1) stabilises

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u}, \quad \hat{x}(0) = \hat{x}_0, \tag{3}$$

where

$$\hat{A} = \mathrm{diag}(A_1, \ldots, A_N), \quad \hat{B} = \mathrm{diag}(B_1, \ldots, B_N), \tag{4}$$

and (2) couples the dynamic behaviour of the agents minimising a cost function which has a weighted norm of $\sum_{i=1}^{N}\sum_{j|j\in\mathcal{N}_i}(x_i - x_j)$ as one of its terms. Meeting these objectives can be formulated as the solution of the following infinite-horizon distributed optimal control problem:

$$\mathcal{P} : \underset{\hat{u}}{\text{minimise}} \quad J(\hat{u}, \hat{x}_0) = \int_0^\infty \left(\hat{x}'\hat{Q}\hat{x} + \hat{u}'\hat{R}\hat{u}\right) \mathrm{d}t \tag{5a}$$

$$\text{subject to} \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u},\ \hat{x}(0) = \hat{x}_0 \tag{5b}$$

$$\hat{u} = \mathcal{M}\hat{x} \tag{5c}$$

$$\mathcal{M} \in \mathcal{K}_{m,n}^N(\mathcal{G}) \tag{5d}$$

$$\hat{Q} = I_N \otimes Q_1 + \mathcal{L} \otimes Q_2 \tag{5e}$$

$$\hat{R} = I_N \otimes R. \tag{5f}$$

Matrices $Q_1 = Q_1' \geq 0$ and $R = R' > 0$ above penalise individual states and inputs, respectively, while matrix $Q_2 = Q_2' \geq 0$ weighs the relative state-difference between neighbours. Note that the objective function (5a) can be expressed in the following form:

$$J(\hat{u},\ \hat{x}_0) = \int_0^\infty \sum_{i=1}^{N} \Bigg(x_i'Q_1x_i + u_i'Ru_i + 0.5 \sum_{j|j\in\mathcal{N}_i} (x_i - x_j)'Q_2(x_i - x_j) \Bigg) \mathrm{d}t. \tag{6}$$

Note also that due to penalty terms pertinent to weighting matrix Q_2, problem (5) couples the dynamics of the (open-loop decoupled) agents. In the absence of constraint (5d) (and under Assumption 3.1), problem (5) is a standard LQR problem which admits a unique stabilising solution $\hat{u} = K^*\hat{x}$ with $K^* = -\hat{R}^{-1}\hat{B}'P^*$ where P^* is the (unique) symmetric positive definite solution to the Algebraic Riccati Equation (ARE):

$$\hat{A}'P^* + P^*\hat{A} - P^*\hat{B}\hat{R}^{-1}\hat{B}'P^* + \hat{Q} = 0. \tag{7}$$

In this instance, matrices P^*, K^*, as defined above, are generically full (with no particular structure) and thus, controller K^* can only be used in a centralised setting. Although, this solution has no value for distributed control, it can be used as a benchmark for testing the performance of suboptimal solutions. For example, if $\hat{K} \in \mathcal{K}_{m,n}^N(\mathcal{G})$ achieves a performance index $V(\hat{x}_0) = \hat{x}_0'\hat{P}\hat{x}_0$, then a norm of $\Delta P = \hat{P} - P^*$ can be used to quantify its level of suboptimality.

Here, rather than seeking the optimal solution to (5), which is an NP-hard problem, we convert problem (5) into two tractable tasks and design a suboptimal distributed state-feedback controller via a two-stage procedure. Using Assumption 3.1-(A4), a model-matching problem for systems $(A_i,\ B_i)$, $i = 1, \ldots, N$ is solved first. The solution involves the selection of a coordinate transformation $x_i \to P_ix_i$, the design of a state-feedback matrix F_i, and an input scaling matrix G_i, such that:

$$P_i(A_i + B_iF_i)P_i^{-1} = A, \tag{8a}$$

$$P_iB_i = B, \tag{8b}$$

for each $i = 1, \ldots, N$. We refer to the pair $(A,\ B)$ in (8) above as the *target system*. See Theorem 4.1 for the derivation of (8). Under the matching transformation (8), the input-to-state part of systems $(A_i,\ B_i)$, $i = 1, \ldots, N$, is mapped to target model $(A,\ B)$. The matching problem is fully analysed in Section 4. Specifically, it is shown that the target model $(A,\ B)$ can be selected arbitrarily from the set of models with the same controllability indices of the family of agents $\{(A_i,\ B_i)\}_{i=1}^N$. We may also introduce an optimality criterion in the selection of the target system by minimising the joint model-matching control effort in a sense made precise in Section 4.6. In this way, the closed-loop properties of the network can be tuned via a network-wide distributed LQR control scheme, carried out in the second stage of the design and involving the minimisation of (5a) subject to:

$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes B)\hat{u},\ \hat{x}(0) = \hat{x}_0, \tag{9a}$$

$$\hat{u} = \mathcal{M}\hat{x}, \tag{9b}$$

$$\mathcal{M} \in \mathcal{K}_{m,n}^N(\mathcal{G}). \tag{9c}$$

Note that after the first stage of the design the dynamics of all agents (with local control) are mapped to $(A,\ B)$ and, hence, any algorithm assuming identical agent dynamics is applicable. Two complementary methods for approximating the optimal solution to problem (5a) subject to (9) are proposed in Borrelli and Keviczky (2008) and Deshpande et al. (2012), respectively. In summary, our method results in a two-level distributed control scheme, with the inner loop being responsible for *matching* agent dynamics and the outer loop for controlling the overall performance of the network. A schematic example of the proposed control architecture is shown in Figure 1. All parameters appearing in Figure 1 will be clearly defined in subsequent sections of the paper.

3.1 Review of distributed LQR control for identical linear systems

Two well-established results in the context of distributed LQR control are reviewed. Both methods rely on the simplifying assumption of identical systems' dynamics. We refer to the control design established in Borrelli and Keviczky (2008) as the

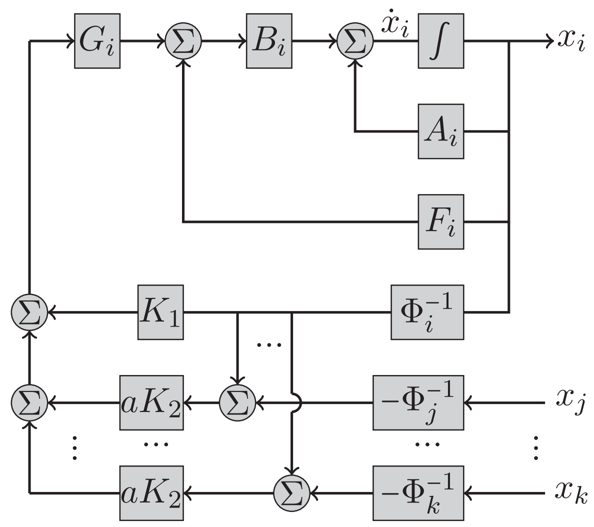

Figure 1. Distributed node-level closed-loop architecture of interconnected heterogeneous linear agents.

top-down method and to the control design established in Deshpande et al. (2012) as the *bottom-up* method, respectively. The methods propose a suboptimal solution to problem (5) where systems matrices in (5b) are considered identical, i.e. $\hat{A} = I_N \otimes A$, $\hat{B} = I_N \otimes B$. The main points of the methods are highlighted in Appendices 1 and 2.

4. Model-matching methods

The top-down and bottom-up methods (cf. Appendices 1 and 2, respectively) are powerful for stabilising homogeneous multi-agent networks via distributed state-feedback control, yet they both rely on the assumption of identical agents which may be unrealistic. In this work we bypass this limitation and extend the methods to a significantly larger class of systems, characterised by the same set of controllability indices. This constitutes a natural assumption, consistent with parametric families of agents sharing the same structural properties.

The model-matching problem is defined as the task of matching the dynamics of a set of heterogeneous linear agents by means of state-feedback control, input matrix transformations and a change of coordinates. This is possible if and only if the agents share a set of controllability indices. Solving the model-matching problem is the first stage of the stabilisation of the network. It is shown that if the model-matching problem is solvable, then the intricate problem of stabilising a network of heterogeneous agents reduces to the special case of homogeneous-agents which is simpler to analyse. It is shown in the sequel that our method can further be extended to the solution of the non-linear version of the problem.

4.1 Problem definition

Problem 4.1 (Model-matching): Consider $N+1$ controllable multi-input linear systems described by the state-space equations:

$$\dot{x}_i = A_i x_i + B_i u_i,\ x_i(0) = x_{i,0},\ i = 1, \ldots, N, \quad (10)$$

with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(B_i) = m$. Let the pair (A_{N+1}, B_{N+1}) pertain to the target dynamics. Then, we wish to find matrices P_i, F_i, and G_i of appropriate dimensions with $\det(P_i) \neq 0$ and $\det(G_i) \neq 0$ such that:

$$P_i(A_i + B_iF_i)P_i^{-1} = A_{N+1} \quad \text{and} \quad P_iB_iG_i = B_{N+1} \quad (11)$$

for $i = 1, \ldots, N$.

Problem 4.1 involves the control design $u_i = F_ix_i + G_iv_i$, $v_i \in \mathbb{R}^m$, $i = 1, \ldots, N$, whereby N systems match their dynamics with a target model denoted as (A_{N+1}, B_{N+1}). It also involves finding matrices P_i, $i = 1, \ldots, N$, which represent similarity transformations accommodating a change of local coordinates. In the following, the class of systems with common controllability indices is defined. It will be shown that a solution to Problem 4.1 is always guaranteed for this family of systems. Some basic concepts which are useful in our definitions and proofs are introduced next.

4.2 Controllability indices of multi-input systems

Recall the notion of controllability indices of a controllable system (A, B): Let $\dot{x} = Ax + Bu, x(0) = x_0$, be the state-space form of a controllable system (A, B), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, with $\mathrm{rank}(B) = m$. Let also

$$\begin{aligned}\mathcal{C} &= [B, AB, \ldots, A^{n-1}B],\\ &= [b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, \ldots, A^{n-1}b_1, \ldots, A^{n-1}b_m],\end{aligned} \quad (12)$$

be the controllability matrix of the pair (A, B), where $b_1, \ldots, b_m$ represent the columns of B, and $\mathcal{C} \in \mathbb{R}^{n\times nm}$. Since (A, B) is controllable, $\mathrm{rank}(\mathcal{C}) = n$. Now, collect the first n linearly independent columns of $\mathcal{C}$ starting from the left and moving to the right; rearrange these columns to obtain

$$\bar{\mathcal{C}} = [b_1, Ab_1, \ldots, A^{\mu_1-1}b_1, \ldots, b_m, Ab_m, \ldots, A^{\mu_m-1}b_m], \quad (13)$$

where $\bar{\mathcal{C}} \in \mathbb{R}^{n\times n}$. The integer μ_j denotes the number of columns involving b_j in the set of the first n linearly independent columns of $\mathcal{C}$ while moving from left to right. The set of μ_j's is defined next.

Definition 4.1: The set of m integers $\{\mu_1, \ldots, \mu_m\}$, as defined in (13), with $\sum_{j=1}^m \mu_j = n$, represents the controllability indices of the controllable pair (A, B) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(B) = m$.

Here, we are interested in distributed stabilising state-feedback solutions to the regulation problem (5) for networks of non-identical agents with identical sets of controllability indices. The following lemma is standard and is included without proof (Antsaklis & Michel, 2006).

Lemma 4.1: *Given (A, B) is controllable, then $(P(A+BF)P^{-1}, PBG)$ has the same controllability indices (c.i.), up to reordering, for any P, F, and G ($\det(P) \neq 0$, $\det(G) \neq 0$) of appropriate dimensions.*

Lemma 4.1 states that the c.i. of a controllable pair $(A,\ B)$ is an invariant set under a state-space transformation P, a state-feedback control F, and an input scaling G. Pertaining to a set of systems $(A_i,\ B_i)$, $i = 1, \ldots, N$, characterised by identical sets of c.i., Lemma 4.1 also implies that pairs $(P_i(A_i + B_iF_i)P_i^{-1},\ P_iB_iG_i)$, $i = 1, \ldots, N$, coincide, for a certain choice of P_i, F_i, and G_i, $i = 1, \ldots, N$. This is clarified in the following section.

4.3 Model-matching: existence conditions

We consider N systems with dynamics described by the equations:

$$\dot{x}_i = A_ix_i + B_iu_i, x_i(0) = x_{i,0}, i = 1, \ldots, N, \tag{14}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are the states and inputs of the ith system, respectively. Let $\mu_1, \ldots, \mu_m$ be the controllability indices of the pairs $(A_i,\ B_i)$, $i = 1, \ldots, N$. Let also P_i be the similarity transformation that brings $(A_i,\ B_i)$ into controllable canonical form. We refer readers to Antsaklis and Michel (2006) (Chapter 3, Section 3.4), for how to construct matrix P_i. Changing coordinates to $x_{c,i} = P_ix_i$, we get

$$\dot{x}_{c,i} = A_{c,i}x_{c,i} + B_{c,i}u_i, \quad x_{c,i}(0) = P_ix_{i,0}, \tag{15}$$

where $x_i = P_i^{-1}x_{c,i}$ is the state vector x_i in the original coordinates. Matrices $A_{c,i}$, $B_{c,i}$ can be decomposed as follows:

$$A_{c,i} = \bar{A}_c + \bar{B}_cA_{m,i}, \quad B_{c,i} = \bar{B}_cB_{m,i}, \tag{16}$$

with $\bar{A}_c \in \mathbb{R}^{n\times n}$, $\bar{B}_c \in \mathbb{R}^{n\times m}$, $A_{m,i} \in \mathbb{R}^{m\times n}$ and $B_{m,i} \in \mathbb{R}^{m\times m}$. The pair $(\bar{A}_c,\ \bar{B}_c)$ is called the Brunovsky canonical form (Antsaklis & Michel, 2006) and is unique for all systems with identical sets of controllability indices. Matrices $(A_{m,i},\ B_{m,i})$ are free; later, we show that the selection of a target model depends on the choice of these two matrices. The Brunovsky form $(\bar{A}_c,\ \bar{B}_c)$ has block-diagonal structure:

$$\bar{A}_c = \mathrm{diag}(\bar{A}_{11}, \ldots, \bar{A}_{mm}), \quad \bar{B}_c = \mathrm{diag}(\bar{B}_{11}, \ldots, \bar{B}_{mm}) \tag{17}$$

where

$$\bar{A}_{jj} = \begin{bmatrix} 0 & \\ \vdots & I_{\mu_j - 1} \\ 0 & \\ 0 & 0\cdots 0 \end{bmatrix} \in \mathbb{R}^{\mu_j\times\mu_j}, \quad \bar{B}_{jj} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\mu_j}, \tag{18}$$

for $j = 1, \ldots, m$. We note here that the diagonal blocks $\bar{A}_{jj}$ are completely defined by the controllability indices $\mu_1, \ldots, \mu_m$. Consider now a target system $(A_{N+1},\ B_{N+1})$, and assume it has common c.i. with the remaining systems in the set. This implies identical Brunovsky forms for all $N+1$ systems. Without loss of generality, let $(A_{N+1},\ B_{N+1})$ be in canonical form. The state-space form of the target system is written as

$$\dot{x}_{N+1} = A_{N+1}x_{N+1} + B_{N+1}u_{N+1} \tag{19}$$

where

$$A_{N+1} = \bar{A}_c + \bar{B}_cA_{m,N+1}, \quad B_{N+1} = \bar{B}_cB_{m,N+1}. \tag{20}$$

The pair $(\bar{A}_c,\ \bar{B}_c)$ represents the Brunovsky form with c.i. $\mu_1, \ldots, \mu_m$, while matrices $A_{m,N+1}$, $B_{m,N+1}$ are as defined earlier. From (16) and (20) it follows that matching $(A_{c,i},\ B_{c,i})$, $i = 1, \ldots, N$, with $(A_{N+1},\ B_{N+1})$ depends exclusively on matrices $A_{m,i}$, $B_{m,i}$, $i = 1, \ldots, N+1$. It is also clear that Problem 4.1 has a solution if and only if the $N+1$ systems have identical sets of controllability indices. This is summarised in the following theorem.

Theorem 4.1: *Consider N controllable systems $(A_i,\ B_i)$, with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $\mathrm{rank}(B_i) = m$, $i = 1, \ldots, N$, and state-space form given in* (14). *Let a target system be described by the state-space form:*

$$\dot{x}_{N+1} = A_{N+1}x_{N+1} + B_{N+1}u_{N+1}, \tag{21}$$

and assume that all pairs $(A_i,\ B_i)$, $i = 1, \ldots, N+1$ have identical c.i.'s, $\mu_1, \ldots, \mu_m$. Then, there are matrices F_i, and G_i, defined as

$$F_i = B_{m,i}^{-1}(A_{m,N+1} - A_{m,i})P_i, \quad G_i = B_{m,i}^{-1}B_{m,N+1}, \tag{22}$$

respectively, such that

$$\Phi_i^{-1}(A_i + B_iF_i)\Phi_i = A_{N+1}, \quad \Phi_i^{-1}B_iG_i = B_{N+1}, \tag{23}$$

where $(A_{m,i},\ B_{m,i})$, $i = 1, \ldots, N$, are defined in (16), *pair $(A_{m,N+1}, B_{m,N+1})$ is defined in* (20), *and $\Phi_i = P_i^{-1}P_{N+1}$, $i = 1, \ldots, N$, with $\det(\Phi_i) \neq 0$. Matrices P_i, $i = 1, \ldots, N+1$, represent similarity transformations that bring the systems in controllable canonical form.*

Proof: See Appendix 3 ■

For a family of N systems with identical sets of controllability indices, Theorem 4.1 guarantees the existence of state-feedback gains F_i and input-matrix scaling transformations G_i such that:

$$\dot{x}_i = (A_i + B_iF_i)x_i + B_iG_iv_i, \quad \xi = \Phi_i^{-1}x_i, \tag{24}$$

for all $i = 1, \ldots, N$, where $\Phi_i = P_i^{-1}P_{N+1}$, with P_i, P_{N+1} as defined in the theorem. Since Φ_i in (24) is non-singular, the map between x_i and ξ is one to one. Note also that for identical initial conditions $x_{i,0}$, and controls v_i, $i = 1, \ldots, N$, the output trajectories of (24) coincide for all $i = 1, \ldots, N$, and are denoted as $\xi(t)$. Variable ξ can be identified as the state of the target model:

$$\dot{x}_{N+1} = A_{N+1}x_{N+1} + B_{N+1}u_{N+1}, \quad \xi = x_{N+1}. \tag{25}$$

Note that the transformations defined in (22) represent the model-matching design of N systems with a target model specified a priori. Next, we introduce further existence conditions that are useful for model-matching control synthesis.

4.4 Model-matching control synthesis

Consider a set of N controllable systems $(A_i,\ B_i)$ with

$$A_i = A_o + B_iZ_i, \quad B_i = B_oG_i^{-1}, \tag{26}$$

for $i = 1, \ldots, N$. Here $A_o \in \mathbb{R}^{n\times n}$ is assumed to be a fixed matrix, $Z_i \in \mathbb{R}^{m\times n}$ an arbitrary matrix, and $G_i \in \mathbb{R}^{m\times m}$ an arbitrary and nonsingular matrix for all $i = 1, \ldots, N$. Note that if all pairs $(A_i,\ B_i)$, $i = 1, \ldots, N$, have identical sets of controllability indices, their controllable canonical forms

$$(A_{c,i},\ B_{c,i}) = (P_i A_i P_i^{-1},\ P_i B_i), \quad i = 1, \ldots, N, \tag{27}$$

satisfy condition (26). In this case, $(A_o,\ B_o) = (\bar{A}_c,\ \bar{B}_c)$ represents the Brunovsky form of all pairs $(A_i,\ B_i)$, $i = 1, \ldots, N$ with common controllability indices. Clearly, a possible target pair $(A_{N+1},\ B_{N+1})$ has to satisfy condition (26). The following lemma guarantees the existence of a input matrix transformation that maps B_i, $i = 1, \ldots, N$, matrices to a target (input) matrix denoted as B_{N+1}. In the following, $\mathrm{Im}(\cdot)$ denotes the column-span of a matrix.

Lemma 4.2: *Let matrices $B_i \in \mathbb{R}^{n\times m}$, $i = 1, \ldots, N$, have full-column rank. Then, there exists a matrix $B_o \in \mathbb{R}^{n\times m}$, and square and nonsingular matrices $G_i \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, N$, such that $B_i G_i = B_o\ \forall i$, if and only if $\mathrm{Im}(B_1) = \mathrm{Im}(B_2) = \cdots = \mathrm{Im}(B_N)$.*

Proof: (i) Necessity: Let $\mathrm{Im}(B_i) = \mathcal{X} \subseteq \mathbb{R}^n$ with $\dim(\mathcal{X}) = m$. Then, B_i has a singular value decomposition: $B_i = U\Sigma_i V_i'$, $i = 1, \ldots, N$, with $\mathrm{Im}(B_i) = \mathrm{Im}(U) = \mathcal{X}$ and $U'U = I_m$, $\det(\Sigma_i) \neq 0$, $V_i'V_i = V_iV_i' = I_m$. Define: $G_i = V_i\Sigma_i^{-1}$, $B_o = U$. Then, $B_iG_i = U\Sigma_iV_i'V_i\Sigma_i^{-1} = U = B_o$. (*ii*) Sufficiency is immediate. ■

A solution to the matching problem 4.1 for N systems $(A_i,\ B_i)$, $i = 1, \ldots, N$, with structure as in (26), is given in the following Theorem. First, we define a special class of systems:

Definition 4.2: Let $(A_o, B_o) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, with $\mathrm{rank}(B_o) = m$. Define the set:

$$S(A_o,\ B_o) = \{(A_o + B_oZ,\ B_oG^{-1}) : Z \in \mathbb{R}^{m\times n}, \\ G \in \mathbb{R}^{m\times m} \text{ with } \det(G) \neq 0\}. \tag{28}$$

Theorem 4.2:

(i) *Let $(A_i,\ B_i) \in S(A_o,\ B_o)$, $i = 1, \ldots, N$. Then,* $\mathrm{Im}(B_i) = \mathrm{Im}(B_o)$, $\forall i = 1, \ldots, N$ *and there exist* $X \in \mathbb{R}^{n\times n}$, $X = X' > 0$, $Y_i \in \mathbb{R}^{m\times n}$ *such that*

$$A_iX + B_iY_i - A_jX - B_jY_j = 0, \tag{29}$$

for every pair $(i, j) \in \{1, 2, \ldots, N\}^2$.

(ii) *Conversely, let* $\{(A_i,\ B_i)\}_{i=1}^N$ *be given with* $\mathrm{Im}(B_i) = \mathcal{X} \subseteq \mathbb{R}^n$, $\forall i = 1, \ldots, N$, *and* $\dim(\mathcal{X}) = m$. *Suppose also that* (29) *is true for every pair* $(i, j) \in \{1, 2, \ldots, N\}^2$ *for some* $X \in \mathbb{R}^{n\times n}$, $X = X' > 0$, *and* $\{Y_i\}_{i=1}^N$, $Y_i \in \mathbb{R}^{m\times n}$, $i = 1, \ldots, N$. *Then, there exist matrices* $A_o \in \mathbb{R}^{n\times n}$, *and* $B_o \in \mathbb{R}^{n\times m}$, *with* $\mathrm{Im}(B_o) = \mathcal{X}$, *such that*

$$(A_i,\ B_i) = S(A_o,\ B_o), \tag{30}$$

for all $i \in \{1, \ldots, N\}$.

Note: If (29) holds for $X = X' > 0$ and $\{Y_i\}_{i=1}^N$, then, for all (29), $\exists\ \{F_i\}_{i=1}^N$, $F_i \in \mathbb{R}^{m\times n}$, such that

$$A_i + B_iF_i = A_j + B_jF_j, \tag{31}$$

for every pair $(i, j) \in \{1, 2, \ldots, N\}^2$.

Proof: See Appendix 4. ■

For numerical reasons, we may wish to relax the exact model-matching (29) or to impose additional conditions. The modified problems are formulated in LMI form (Boyd et al., 1994) in the following paragraphs.

4.5 Approximate model-matching and stability constraints

As shown in Theorem 4.2, the model-matching problem $A_i + B_iF_i = A_j + B_jF_j$ can be written as $A_iX + B_iY_i = A_jX + B_jY_j$ where $F_i = Y_iX^{-1}$, $F_j = Y_jX^{-1}$, for $i, j = 1, \ldots, N$, and $X = X' > 0$. For a sufficiently small tolerance $\gamma > 0$ this can be approximated as

$$\|A_iX + B_iY_i - (A_jX + B_jY_j)\| < \gamma, \tag{32}$$

for $i, j = 1, \ldots, N$ and $i \neq j$. We will also make use of the following well-known equivalence:

Lemma 4.3: *Let $\Phi \in \mathbb{R}^{n\times n}$ be an arbitrary matrix. The following are equivalent.*

$$\|\Phi\| < \gamma \Leftrightarrow \Phi'\Phi < \gamma^2 I_n \Leftrightarrow \begin{bmatrix} I_n & \Phi \\ \Phi' & \gamma^2 I_n \end{bmatrix} > 0. \tag{33}$$

Using Lemma 4.3, conditions (32) can be formulated as a set of linear matrix inequalities (LMI's):

$$X = X' > 0,\ \begin{bmatrix} I & A_iX + B_iY_i - A_jX - B_jY_j \\ * & \gamma^2 I \end{bmatrix} \geq 0, \tag{34}$$

for $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$. The system can be solved as a standard LMI feasibility problem for a pre-specified positive tolerance γ. The state feedback matrices can then be set as $F_i = Y_iX^{-1}$. The inertia of $(A_i + B_iF_i)$ can be controlled by imposing additional LMI constraints as shown next.

It may be desirable to solve the model-matching problem so that a stable target model is achieved. In a network setup, this guarantees stability of individual systems even in the presence of communication failure between agents. In this regard, we may wish to assign the poles of the target system inside a specific region of the complex plane. If this region is convex, the constraints can be expressed as LMI's. The region may be selected to ensure a minimum decay rate of the response, a maximum undamped natural frequency, and a minimum damping ratio. These performance parameters are denoted next by λ, ρ, and θ, respectively. A comprehensive analysis of pole assignment via LMI constraints can be found in Chilali and Gahinet (1996).

We consider a set of N controllable pairs $(A_i,\ B_i)$, $i = 1, \ldots, N$, with structure as in (26), and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$. We wish to construct state-feedback matrices F_i, $i = 1, \ldots, N$, so that the eigenvalues of the ith matrix $A_i + B_iF_i$ lie in a convex region of the complex plane defined by performance parameters

λ, ρ, and θ. Let

$$X = X' > 0, \tag{35a}$$

$$\begin{bmatrix} I & A_iX + B_iY_i - A_jX - B_jY_j \\ * & \gamma^2 I \end{bmatrix} \geq 0 \text{ for } i,j = 1,\ldots,N \text{ and } i \neq j, \tag{35b}$$

$$\lambda X + \Lambda_i + \Lambda_i' < 0, i \in \{1,\ldots,N\}, \tag{35c}$$

$$\begin{bmatrix} -\rho X & \Lambda_i' \\ * & -\rho X \end{bmatrix} < 0, i \in \{1,\ldots,N\}, \tag{35d}$$

$$\begin{bmatrix} \sin\theta[\Lambda_i + \Lambda_i'] & \cos\theta[-\Lambda_i + \Lambda_i'] \\ * & \sin\theta[\Lambda_i + \Lambda_i'] \end{bmatrix} < 0, i \in \{1,\ldots,N\}, \tag{35e}$$

where $\Lambda_i = A_iX + B_iY_i$, $i = 1,\ldots,N$, and $\gamma > 0$ is a small tolerance. Solving convex feasibility problem (35) yields a target system with the desirable dynamics.

4.6 Optimal selection of target system

So far, we have shown that the model-matching problem of a family of systems characterised by identical sets of controllability indices can be solved via state-feedback control and state/input-matrix transformations. The target system can be selected arbitrarily provided that it matches the set of controllability indices, see (22). Model-matching controllers satisfying additional objectives (stability, pole location) can also be designed via linear matrix inequalities. In this section, we consider the 'optimal' choice of the target model, obtained by minimising the joint model-matching control effort of the local feedback schemes. We impose this objective because we wish to apply the *minimum amount of feedback* in the first stage of the control design of problem (5), so that overall system properties are effectively determined by the quadratic performance index defined at network level (5a). For this purpose, we introduce a cost-function corresponding to a specific measure of the joint model-matching energy loss whose minimisation results in a specific optimal target model. A worst-case control effort index is first examined, defined as a discrete minimax problem which is solved via a non-smooth steepest-descent algorithm (Boyd & Vandenberghe, 2004; Dem'yanov & Malozemov, 2014). A quadratic cost-function is also used which gives rise to a closed-form expression for the optimal local control.

4.6.1 State-feedback design for optimal target system

We consider a set of N systems represented by controllable pairs $(A_i,\ B_i)$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ with $\text{rank}(B_i) = m$ and state-space forms given as in (14). Let $(A_{N+1},\ B_{N+1})$ be a target model with $A_{N+1} \in \mathbb{R}^{n\times n}$, $B_{N+1} \in \mathbb{R}^{n\times m}$. We assume that systems $(A_i,\ B_i)$, $i = 1,\ldots,N+1$ have identical sets of controllability indices denoted as $\mu_1,\ldots,\mu_m$, with $\sum_{j=1}^m \mu_j = n$. Without loss of generality, let $(A_{N+1},\ B_{N+1})$ be written in canonical form given in (16), as

$$A_{N+1} = \bar{A}_c + \bar{B}_c A_{m,N+1}, \quad B_{N+1} = \bar{B}_c B_{m,N+1}, \tag{36}$$

where $A_{m,N+1} \in \mathbb{R}^{m\times n}$, $B_{m,N+1} \in \mathbb{R}^{m\times m}$ are defined in (16). The pair $(\bar{A}_c,\ \bar{B}_c)$ represents the Brunovsky form of all linear systems with controllability indices $\mu_1,\ldots,\mu_m$. To simplify the subsequent analysis, we consider $B_{m,N+1} = I_m$. Model-matching state-feedback gains and input-matrix transformations defined in (22) are written as

$$F_i = B_{m,i}^{-1}(A_{m,N+1} - A_{m,i})P_i, \quad G_i = B_{m,i}^{-1}, \tag{37}$$

where matrices $A_{m,i}$, $A_{m,N+1}$, $B_{m,i}$, P_i are as defined in Theorem 4.1. Letting now $u_i = F_ix_i + G_iv_i$, $i = 1,\ldots,N$, with $v_i \in \mathbb{R}^m$, the closed-loop state-space form of the ith system

$$\dot{x}_i = (A_i + B_iF_i)x_i + B_iG_iv_i, \tag{38}$$

matches the target dynamics:

$$\dot{\xi} = A_{N+1}\xi + B_{N+1}u_{N+1}, \tag{39}$$

through the bijective mapping $\xi = \Phi_i^{-1}x_i$, where $\Phi_i = P_i^{-1}$. In (37), $A_{m,N+1}$ is the only term that associates a specific target selection with local model-matching control action. Hence, we wish to identify $A_{m,N+1}$ which minimises a measure of the joint model-matching control-effort defined as a function of state-feedback matrices F_i, $i = 1,\ldots,N$. Two cost functions are considered, referred to as model-matching indexes. We first state the following well-known fact which accommodates an isometric embedding of the Frobenius norm of a matrix in $\mathbb{R}^{m\times n}$ into the Euclidean norm of a vector in $\mathbb{R}^{mn}$.

Proposition 4.1: *Consider*

$$J(\Xi) = \|A\Xi B - C\|_F^2, \tag{40}$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{q\times q}$, C, $\Xi \in \mathbb{R}^{p\times q}$. Let $\text{vec}(\cdot)$ denote the vectorisation operator (stacking columns of argument matrix). Then

$$\text{vec}(A\Xi B - C) = (B' \otimes A)\text{vec}(\Xi) - \text{vec}(C). \tag{41}$$

Let also $B' \otimes A = H$, $\text{vec}(C) = c$, and $\text{vec}(\Xi) = \xi$. Since $\|\mathcal{M}\|_F = \|\text{vec}(\mathcal{M})\|$, then

$$J(\Xi) = \|H\xi - c\|^2, \tag{42}$$

where $\|\cdot\|$ is the Euclidean norm.

4.6.2 Minimum worst-case control

We denote the joint worst-case model-matching control action as

$$\phi(A_{m,N+1}) = \max_{i=[1:N]} M_i, \text{ where } M_i = \|F_i\|_F^2 = \|B_{m,i}^{-1}(A_{m,N+1} - A_{m,i})P_i\|_F^2. \tag{43}$$

We wish to find matrix $A_{m,N+1} \in \mathbb{R}^{m\times n}$ for which $\phi(A_{m,N+1})$ attains its minimum. This is a discrete minimax problem formulated as

$$\min_{A_{m,N+1}\in\mathbb{R}^{m\times n}} \phi(A_{m,N+1}) = \min_{A_{m,N+1}\in\mathbb{R}^{m\times n}} \max_{i\in[1:N]} \|B_{m,i}^{-1}(A_{m,N+1} - A_{m,i})P_i\|_F^2. \tag{44}$$

To perform the optimisation over $\mathbb{R}^{mn}$, we utilise the vectorisation technique in Proposition 4.1. Let $M_i = \|H_i\xi -$

$c_i\|^2$ where $\Xi = A_{m,N+1}$, $H_i = P_i' \otimes B_{m,i}^{-1}$, $C_i = B_{m,i}^{-1}A_{m,i}P_i$, $\xi = \mathrm{vec}(\Xi)$, $c_i = \mathrm{vec}(C_i)$, $i = 1, \ldots, N$. The minimax problem (44) becomes

$$\begin{aligned} \min_{\xi \in \mathbb{R}^{m \cdot n}} \phi(\xi) &= \min_{\xi \in \mathbb{R}^{m \cdot n}} \max_{i=[1:N]} M_i \\ &= \min_{\xi \in \mathbb{R}^{m \cdot n}} \max_{i=[1:N]} \xi'(H_i'H_i)\xi - 2\xi'(H_i'c_i) + c_i'c_i. \end{aligned} \tag{45}$$

$\phi(\xi)$ is continuous and convex by the continuity and convexity of M_i, $i = 1, \ldots, N$, and its sub-level sets are bounded. Thus, a minimising solution ξ^* exists and is unique. The ϵ-steepest decent algorithm can be employed to find ξ^*. The optimal solution is derived as $A_{m,N+1}^* = \mathrm{vec}^{-1}(\xi^*)$. Optimal state-feedback gains F_i, $i = 1, \ldots, N$, are constructed by substituting $\bar{A}_m^*$ into (37), while an optimal target system is defined as $(\bar{A}_c + \bar{B}_c A_{m,N+1}^*, \bar{B}_c)$.

4.6.3 Least-squares control

Another measure that penalises the joint model-matching control effort is defined as

$$J(A_{m,N+1}) = \sum_{i=1}^{N} \|F_i\|_F^2 = \sum_{i=1}^{N} \|B_{m,i}^{-1}(A_{m,N+1} - A_{m,i})P_i\|_F^2. \tag{46}$$

Here, we are interested in finding a matrix $A_{m,N+1}$ for which J in (46) becomes minimum. Setting $\Xi = A_{m,N+1}$, $H_i = P_i' \otimes B_{m,i}^{-1}$, $C_i = B_{m,i}^{-1}A_{m,i}P_i$, $\xi = \mathrm{vec}(\Xi)$, $c_i = \mathrm{vec}(C_i)$, $i = 1, \ldots, N$, and embedding each matrix $B_{m,i}^{-1}(A_{m,N+1} - A_{m,i})P_i$ into $\mathbb{R}^{mn}$ as suggested in Proposition 4.1, we can optimise J over $\xi \in \mathbb{R}^{mn}$. This is written as

$$\begin{aligned} J(\xi) &= \sum_{i=1}^{N} \|H_i\xi - c_i\|^2 = \sum_{i=1}^{N} (\xi' H_i' - c_i')(H_i\xi - c_i) \\ &= \xi' \left(\sum_{i=1}^{N} H_i'H_i \right) \xi - 2\xi' \left(\sum_{i=1}^{N} H_i'c_i \right) + \sum_{i=1}^{N} c_i'c_i, \end{aligned} \tag{47}$$

which is a convex function of $\xi \in \mathbb{R}^{mn}$. Setting derivative $J_\xi = 0$ gives

$$A_{m,N+1}^* = \mathrm{vec}^{-1} \left(\sum_{i=1}^{N} (H_i'H_i)^{-1} \sum_{i=1}^{N} H_i'c_i \right). \tag{48}$$

after some algebra. Optimal state-feedback gains F_i, $i = 1, \ldots, N$, are obtained by substituting $A_{m,N+1}^*$ into (37).

4.6.4 Remarks on optimal target model derivation

As shown above the minimum of (46) can be found exactly by the least-squares solution (Boyd & Vandenberghe, 2004), while the approximate solution to (45) can be obtained via standard minimax algorithms (Dem'yanov & Malozemov, 2014). Interested readers are also referred to Vlahakis (2020, Chapter 5) for an efficient ε-steepest descent algorithm, with analytic derivation of step-size, consistent with discrete minimax problems with quadratic costs.

Under the assumption that the topology of the network is time-invariant and the models of the agents are known and fixed, the solution to either version of the optimal model-matching problem can be obtained offline via a centralised algorithm. This formulation is consistent with the design of a distributed LQR network controller which is our ultimate goal. The derivation of the optimal target system via a distributed algorithm would be relevant for a time-varying network topology, e.g. in case agents are repeatedly connected and disconnected from the network. In this case, the optimal target system would also vary with time and could be calculated via a distributed consensus-type algorithm by sharing agents' model information through the network. A detailed analysis of this case is beyond the scope of the present work. Preliminary results in this direction can be found in Vlahakis (2020, Chapter 9).

4.7 Nonlinear systems

Motivated by the feedback linearisation technique (Vidyasagar, 2002), in this section we show that the model-matching control protocol can be readily extended to a class of systems with nonlinear dynamics. Thus, in a network setup of self-linearisable agents, mapped to a linear target model via nonlinear model-matching techniques, the regulation problem can be solved via linear LQR-based control. Here we highlight the method and its advantages. A more detailed exposition will be presented in a future publication.

4.7.1 Nonlinear model-matching

We consider a set of N nonlinear systems of the form

$$\dot{x}_i = \mathbf{f}_i(x_i) + \sum_{j=1}^{m} u_{i,j}\mathbf{g}_{i,j}(x_i), \; x_i(0) = x_{i,0}, \quad i = 1, \ldots, N, \tag{49}$$

where $\mathbf{f}_i$, $\mathbf{g}_{i,1}, \ldots, \mathbf{g}_{i,m}$ are smooth vector fields on some neighbourhood $X_i \subseteq \mathbb{R}^n$ near the origin containing $x_{i,0}$, with $\mathbf{f}_i(\mathbf{0}) = \mathbf{0}$. We assume that vector fields $\mathbf{g}_{i,1}, \ldots, \mathbf{g}_{i,m}$, $i = 1, \ldots, N$, are linearly independent for all $x_i \in X_i$. In the following, we denote the input vectors as

$$u_i = \begin{bmatrix} u_{i,1} & \cdots & u_{i,m} \end{bmatrix}', \quad i = 1, \ldots, N. \tag{50}$$

Let

$$\dot{\xi} = A\xi + Bv, \tag{51}$$

be a (linear) target system, where (A, B) is a controllable pair with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$. Without loss of generality, let (A, B) be in controllable canonical form, i.e.

$$A = \bar{A}_c + \bar{B}_c A_m, \quad B = \bar{B}_c B_m, \tag{52}$$

where pair $(\bar{A}_c, \bar{B}_c)$ denotes the Brunonvsky canonical form associated with the set of controllability indices: $\mu_1, \ldots, \mu_m$, with $\sum_{j=1}^{m} \mu_j = n$ (cf. (17)) and $A_m \in \mathbb{R}^{m \times n}$, $B_m = \mathbb{R}^{m \times m}$ with $\det(B_m) \neq 0$. We now summarise the model-matching task of N (feedback linearisable) nonlinear systems as follows.

Problem 4.2 (Nonlinear model-matching)**:** Consider N multi-input nonlinear systems as in (49), and a target system defined by a controllable pair (A, B) as in (52). Letting $U_i \subseteq X_i$ denote a neighbourhood nearby the origin with $x_{i,0} \in U_i$, $i = 1, \ldots, N$, we wish to compute:

(a) a smooth function $\mathbf{q}_i : U_i \to \mathbb{R}^m$ for each $i = 1, \ldots, N$,
(b) a smooth function $\mathbf{S}_i : U_i \to \mathbb{R}^{m\times m}$ such that $\det(\mathbf{S}_i(x_i)) \neq 0 \; \forall x_i \in U_i$, for each $i = 1, \ldots, N$,
(c) a local smooth diffeomorphism $T_i : U_i \to \mathbb{R}^n$, with $T_i(\mathbf{0}) = \mathbf{0}$, for each $i = 1, \ldots, N$,
(d) a state-feedback gain matrix $F_i \in \mathbb{R}^{m\times n}$, for each $i = 1, \ldots, N$,

satisfying the following conditions: if we define feedback control

$$u_i = -\mathbf{S}_i^{-1}(x_i)\mathbf{q}_i(x_i) + \mathbf{S}_i^{-1}(x_i)F_iT_i(x_i) + \mathbf{S}_i^{-1}(x_i)B_m\hat{v}_i, \quad (53)$$

with $\hat{v}_i \in \mathbb{R}^m$ and perform a change of coordinates $z_i = T_i(x_i)$, $i = 1, \ldots, N$, then $\dot{z}_i = Az_i + B\hat{v}_i$, $i = 1, \ldots, N$, where $(A,\ B)$ denotes a target model.

Remark 4.1: The nonlinear transformations and feedback functions considered in Problem 4.2 are defined locally, i.e. in a neighbourhood of initial states defined as the set of permissible states. Thus, a nonlinear model-matching control scheme is not generically valid for all possible initial conditions. This represents a fundamental difference to the linear model-matching where the results are global irrespective of system states.

Suppose that functions $\mathbf{S}_i(x_i)$, $\mathbf{q}_i(x_i)$, $T_i(x_i)$, exist $\forall i = 1, \ldots, N$. Then, necessary and sufficient conditions for Problem 4.2 to have a solution are given in the following theorem.

Theorem 4.3: *Given N (feedback linearisable) systems of the form* (49), *each associated with a set of integers $\kappa_1^i, \ldots, \kappa_m^i$ constructed as in Procedure A.1 (cf. Appendix 5), and a linear target system $(A,\ B)$ as defined in* (51) *pertinent to Brunovsky form $(\bar{A}_c,\ \bar{B}_c)$ associated with controllability indices $\mu_1, \ldots, \mu_m$, Problem* 4.2 *has a solution if and only if the following condition is satisfied:*

(i) the sets $\{\kappa_1^i, \ldots, \kappa_m^i\}$ and $\{\mu_1, \ldots, \mu_m\}$ coincide for all $i = 1, \ldots, N$.

Proof: Detailed proof can be found in Vlahakis (2020, Chapter 6). ■

5. Distributed LQR-based control design

The section describes the second stage of the proposed distributed control design procedure. We recall that our objective is to construct a stabilising distributed solution to problem (5) following a two-step design method. In particular, we propose a state-feedback distributed control scheme which, node-wise, takes the following form:

$$u_i = (F_i + G_iK_1\Phi_i^{-1})x_i + aG_i\sum_{j\in\mathcal{N}_i} K_2(\Phi_i^{-1}x_i - \Phi_j^{-1}x_j),$$
$$i = 1, \ldots, N, \quad (54)$$

where $a > 0$. At network level, the control law $\hat{u}$ may be written as

$$\hat{u} = \big(\mathrm{diag}(F_1, \ldots, F_N) + \mathrm{diag}(G_1, \ldots, G_N)(I_N \otimes K_1 + M \otimes K_2)\mathrm{diag}(\Phi_1^{-1}, \ldots, \Phi_N^{-1})\big)\,\hat{x}, \quad (55)$$

which is a distributed state-feedback controller. Matrix $M \in \mathcal{K}_{1,1}^N(\mathcal{G})$ in (55) is associated with the structure of the interconnection scheme. For control scheme (55) to be consistent with the node-level controller in (54), matrix $M = a\mathcal{L}$, with $a > 0$.

In the first stage of the proposed design, we showed that solving a model-matching problem as suggested in Theorem 4.1, matrices F_i, G_i and Φ_i, $i = 1, \ldots, N$ can be constructed such that:

$$\dot{x}_i = (A_i + B_iF_i)x_i + B_iG_iv_i,\ x_i(0) = x_{i,0},\ \xi_i = \Phi_i^{-1}x_i, \quad (56)$$

where Φ_i, $i = 1, \ldots, N$, is a non-singular matrix, while the output variable $\xi_i \in \mathbb{R}^n$ can be cast as the state of a system described by target dynamics:

$$\dot{\xi} = A\xi + Bv. \quad (57)$$

Given that a target system $(A,\ B)$ has been specified, the next step of the method is to define matrices K_1, K_2, and M in (55), such that $\hat{u}$ stabilises (5b), i.e. is a stabilising distributed (suboptimal) solution to problem (5). This represents the main task of the second stage of our control design approach and is outlined next.

The closed-loop state-space forms (56) match the dynamics of the target model $(A,\ B)$ through a bijective mapping represented by matrices Φ_i^{-1}, $i = 1, \ldots, N$. Therefore, matrices K_1, K_2 can be defined as functions of $(A,\ B,\ Q_1,\ Q_2,\ R)$, where pair $(A,\ B)$ denotes the model-matching dynamics, while matrices $(Q_1,\ Q_2,\ R)$ are tuning parameters of (5a). The design matrices K_1 and K_2 can now be derived from the methods presented in Appendices 1 and 2. This simplifies considerably the design procedure and highlights the main advantage of the model-matching technique proposed in the paper. The proposed control design is summarised in the following theorem.

Theorem 5.1: *Let N controllable pairs $(A_i,\ B_i)$, $i = 1, \ldots, N$, with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, have identical sets of controllability indices. Let also matrices A, B, F_i, G_i, and Φ_i be defined as in Theorem* 4.1, *such that*

$$(A,\ B) = (\Phi_i^{-1}(A_i + B_iF_i)\Phi_i,\ \Phi_i^{-1}B_iG_i),\ i = 1, \ldots, N, \quad (58)$$

where $(A,\ B)$ is a controllable pair. Consider LQR problem (A2) *with tuning parameters $(Q_1,\ Q_2,\ R)$ for $N_L = d_{\max} + 1$ identical systems with dynamics described by the pair $(A,\ B)$, $d_{\max}$ denoting the maximum vertex degree of the associated graph. Define matrices P, $\tilde{P}_2$ as in* (A5), (A6), *respectively, and let $K_1 = -R^{-1}B'P$, $K_2 = R^{-1}B'\tilde{P}_2$. Let also $M \in \mathbb{R}^{N\times N}$ be a symmetric matrix with the following property:*

$$\lambda_i(M) > \frac{N_L}{2}, \quad \forall\, \lambda_i(M) \in S(M)\backslash\{0\}, \quad (59)$$

and construct a state-feedback controller as

$$\hat{K} = \mathrm{diag}(F_1, \ldots, F_N) + \mathrm{diag}(G_1, \ldots, G_N)(I_N \otimes K_1 + M \otimes K_2)\mathrm{diag}(\Phi_1^{-1}, \ldots, \Phi_N^{-1}). \quad (60)$$

Then, the closed-loop matrix

$$A_{cl} = \mathrm{diag}(A_1, \ldots, A_N) + \mathrm{diag}(B_1, \ldots, B_N)\hat{K}, \quad (61)$$

is Hurwitz.

Proof: We define matrix $\bar{\Phi}$ as: $\bar{\Phi} = \mathrm{diag}(\Phi_1, \ldots, \Phi_N)$. From Theorem 4.1, matrices Φ_i, $i = 1, \ldots, N$, are nonsingular, hence $\bar{\Phi}$ is also nonsingular. Equation (58) implies

$$\bar{\Phi}(I_N \otimes A)\bar{\Phi}^{-1} = \mathrm{diag}(A_1 + B_1F_1, \ldots, A_N + B_NF_N), \quad (62)$$

$$\bar{\Phi}(I_N \otimes B) = \mathrm{diag}(B_1G_1, \ldots, B_NG_N). \quad (63)$$

Then, expanding A_{cl} as

$$A_{cl} = \underbrace{\mathrm{diag}(A_1 + B_1F_1, \ldots, A_N + B_NF_N)}_{a_1} + \underbrace{\mathrm{diag}(B_1G_1, \ldots, B_NG_N)}_{b_1}(I_N \otimes K_1 + M \otimes K_2)\bar{\Phi}^{-1}, \quad (64)$$

and substituting a_1 and b_1 using (62) and (63), respectively, (64) becomes

$$A_{cl} = \bar{\Phi}\,(I_N \otimes A + (I_N \otimes B)(I_N \otimes K_1 + M \otimes K_2))\,\bar{\Phi}^{-1}. \quad (65)$$

From (65), matrices A_{cl}, $I_N \otimes A + (I_N \otimes B)(I_N \otimes K_1 + M \otimes K_2)$ are similar. Also, from Theorem A.1, $I_N \otimes A + (I_N \otimes B)(I_N \otimes K_1 + M \otimes K_2)$ is a Hurwitz matrix which implies that the closed-loop matrix A_{cl} is also Hurwitz. This proves the theorem. ■

The main consequences of Theorem 5.1 are summarised as follows:

(1) The state-feedback controller $\hat{K}$ in (60) has a distributed sparsity pattern since $M \in \mathcal{K}^N_{1,1}(\mathcal{G})$.
(2) We can use any method for designing matrices K_1, K_2, and M that guarantees stability of the closed-loop system $I_N \otimes A + (I_N \otimes B)(I_N \otimes K_1 + M \otimes K_2)$.
(3) In the setting of Theorem 5.1, closed-loop stability of the distributed scheme holds irrespective of the tuning of the LQR performance index. Thus, by minimising the joint model-matching energy loss, network's performance is effectively controlled by tuning parameters Q_1, Q_2 and R.

6. Numerical example: stabilisation of network of non-identical oscillators

We consider a network of eleven harmonic oscillators, each one modelled by a two-mass-two-spring system as depicted in Figure 2. The ith oscillator is composed of two masses, $m_{i,1}$ and $m_{i,2}$, which are connected through a spring with spring constant $k_{i,2}$, with mass $m_{i,1}$ attached to a rigid object through a spring with spring constant $k_{i,1}$. Input forces $u_{i,1}$ and $u_{i,2}$ are applied to $m_{i,1}$ and $m_{i,2}$, respectively. The displacement of the two masses from their equilibrium position is denoted as $x_{i,1}$ and $x_{i,2}$, respectively, $i = 1, \ldots, 11$. The models of the eleven oscillators are:

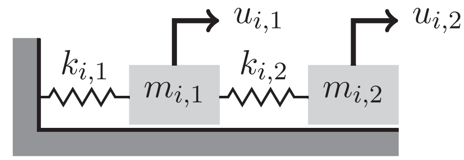

Figure 2. Two-mass-two-spring harmonic oscillator.

$$\begin{bmatrix} \dot{x}_{i,1} \\ \ddot{x}_{i,1} \\ \dot{x}_{i,2} \\ \ddot{x}_{i,2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_{i,1} - k_{i,2}}{m_{i,1}} & 0 & \frac{k_{i,2}}{m_{i,1}} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_{i,2}}{m_{i,2}} & 0 & \frac{-k_{i,2}}{m_{i,2}} & 0 \end{bmatrix}}_{A_i} \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \dot{x}_{i,1} \\ \dot{x}_{i,2} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{m_{i,1}} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_{i,2}} \end{bmatrix}}_{B_i} \begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix}, \quad i = 1, \ldots, 11. \quad (66)$$

Parameters of all oscillators are summarised in Table 1. State-space forms (66) are clearly in controllable canonical form. Thus matrices $(A_i,\ B_i)$ may be written as

$$A_i = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\bar{A}_c} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\bar{B}_c} \underbrace{\begin{bmatrix} \frac{-k_{i,1} - k_{i,2}}{m_{i,1}} & 0 & \frac{k_{i,2}}{m_{i,1}} & 0 \\ \frac{k_{i,2}}{m_{i,2}} & 0 & \frac{-k_{i,2}}{m_{i,2}} & 0 \end{bmatrix}}_{A_{m,i}}, \quad i = 1, \ldots, 11, \quad (67a)$$

$$B_i = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\bar{B}_c} \underbrace{\begin{bmatrix} \frac{1}{m_{i,1}} & 0 \\ 0 & \frac{1}{m_{i,2}} \end{bmatrix}}_{B_{m,i}}, \quad i = 1, \ldots, 11, \quad (67b)$$

respectively. The pair $(\bar{A}_c,\ \bar{B}_c)$ denotes the Brunovsky form which is identical to all systems $(A_i,\ B_i)$, $i = 1, \ldots, 11$. Systems $(A_i,\ B_i)$, $i = 1, \ldots, 11$, have identical controllability indices which are identified here as $\mu_1 = 2$ and $\mu_2 = 2$.

We use a graph representation of the interaction between neighbouring oscillators. The interconnection scheme considered in simulations is shown in Figure 3. The associated graph indicates that if an edge $(i,\ j)$, $i,\ j = 1, \ldots, 11$, $i \neq j$ is present,

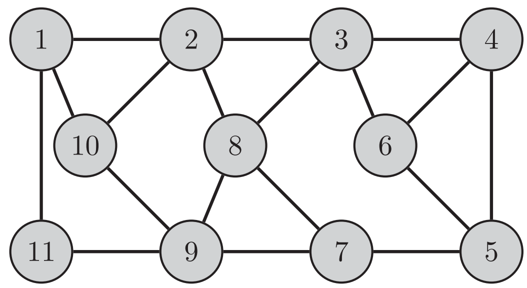

Figure 3. Graph $\mathcal{G}_{11}$: Interconnection scheme of eleven oscillators.

Table 1. Masses and spring constants.

System	$k_{i,1}$	$k_{i,2}$	$m_{i,1}$	$m_{i,2}$
oscillator 1	1.50 N/m	1.00 N/m	1.10 kg	0.90 kg
oscillator 2	3.10 N/m	2.00 N/m	2.10 kg	1.50 kg
oscillator 3	0.50 N/m	1.10 N/m	1.50 kg	3.20 kg
oscillator 4	2.00 N/m	1.30 N/m	3.10 kg	2.10 kg
oscillator 5	1.70 N/m	3.10 N/m	4.10 kg	2.50 kg
oscillator 6	2.20 N/m	4.20 N/m	5.10 kg	4.20 kg
oscillator 7	4.10 N/m	2.50 N/m	1.20 kg	5.10 kg
oscillator 8	2.50 N/m	1.80 N/m	5.10 kg	2.30 kg
oscillator 9	10.5 N/m	30.3 N/m	1.30 kg	1.20 kg
oscillator 10	2.70 N/m	0.80 N/m	1.40 kg	5.20 kg
oscillator 11	5.20 N/m	2.20 N/m	3.50 kg	2.40 kg

then the i th oscillator is aware of the state of the jth oscillator and vice versa. We denote the Laplacian matrix of graph $\mathcal{G}_{11}$ by $\mathcal{L}_{G,11}$.

In the study, we wish to regulate the mass displacement of eleven oscillators given arbitrary initial conditions and the interconnection topology shown in Figure 3. This task can be resolved by solving the following regulator problem:

$$\min_{\hat{u}} J(\hat{u}, \hat{x}_0) \text{ subject to:} \tag{68a}$$

$$J(\hat{u}, \hat{x}_0) = \int_0^\infty \left(\hat{x}'\hat{Q}\hat{x} + \hat{u}'\hat{R}\hat{u}\right) \mathrm{d}t \tag{68b}$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u}, \hat{x}(0) = \hat{x}_0 \tag{68c}$$

$$\begin{aligned}\hat{u} = {} & (\mathrm{diag}(F_1, \dots, F_N) \\ & + \mathrm{diag}(G_1, \dots, G_N)(I_N \otimes K_1 \\ & + M \otimes K_2)\mathrm{diag}(\Phi_1^{-1}, \dots, \Phi_N^{-1}))\hat{x},\end{aligned} \tag{68d}$$

where

$$\hat{A} = \mathrm{diag}(A_1, \dots, A_N), \quad \hat{B} = \mathrm{diag}(B_1, \dots, B_N), \tag{69}$$

and

$$\hat{Q} = I_N \otimes Q_1 + \mathcal{L}_{G,11} \otimes Q_2, \quad \hat{R} = I_N \otimes R. \tag{70}$$

We follow a two-stage control design as suggested earlier in the paper. First, we construct the model-matching design parameters $F_i,\ G_i,\ \Phi_i,\ i = 1, \dots, 11$. Since systems $(A_i,\ B_i),\ i = 1, \dots, 11$, are in controllable canonical form, matrices $\Phi_i,\ i = 1, \dots, 11$, can automatically be selected as the identity matrix. Setting $G_i = B_{m,i}^{-1},\ i = 1, \dots, 11$, we design model-matching state-feedback matrices $F_i,\ i = 1, \dots, 11$, in the following fashion. We consider two model-matching performance indexes represented by

$$J_{\mathrm{aver}}(A_m) = \sum_{i=1}^N \|F_i\|_F^2 = \sum_{i=1}^N \|B_{m,i}^{-1}(A_m - A_{m,i})\|_F^2, \tag{71}$$

and

$$J_{\max}(A_m) = \max_{i\in[1:11]} \|F_i\|_F^2 = \max_{i\in[1:11]} \|B_{m,i}^{-1}(A_m - A_{m,i})\|_F^2, \tag{72}$$

respectively. Obviously, minimising these two cost functions over matrices in $\mathbb{R}^{2\times 4}$ results in two different target models which are used later in the simulations. We denote by $A_{m,\mathrm{aver}}^*$ the minimiser of $J_{\mathrm{aver}}(\cdot)$ and by $A_{m,\max}^*$ the approximate minimiser of $J_{\max}(\cdot)$. The least-squares solution and the minimax solution (obtained from MatLab function `fminimax` with convergence error tolerance set to 10^{-5}) were obtained as

$$A_{m,\mathrm{aver}}^* = \begin{bmatrix} -1.8534 & 0 & 1.0178 & 0 \\ 0.9168 & 0 & -0.9168 & 0 \end{bmatrix},$$
$$A_{m,\max}^* = \begin{bmatrix} -7.7488 & 0 & 5.6556 & 0 \\ 6.7050 & 0 & -6.7050 & 0 \end{bmatrix} \tag{73}$$

respectively. Note the relatively large distance between matrices $A_{m,\mathrm{aver}}^*$ and $A_{m,\max}^*$. This stems from the extreme choice of high springs' stiffness of oscillator 9 as seen in Table 1. In essence, the least-squares solution $A_{m,\mathrm{aver}}^*$ attempts to achieve the average of $\|F_i\|_F^2,\ i = 1, \dots, 11$, while the approximate minimiser $A_{m,\max}^*$ is clearly attracted from the outlier matrix $A_{m,9}$. The optimal model of the target system arising from the minimisation of the two performance indexes above is obtained from $(\bar{A}_c + \bar{B}_c A_m^*,\ \bar{B}_c)$, A_m^* denoting each particular optimal solution shown in (73) respectively.

An alternative target model selection is outlined next. As mentioned in Section 4, it is possible to achieve a target model with certain performance specifications. As a third target choice, we require the eigenvalues of the target system lie in the cone represented by two line segments starting from the origin, each line segment forming an angle of $\pi/3$ with the negative real axis. Requiring $\lambda = 0, \rho = 0, \theta = \pi/3$ and minimising parameter γ subject to (35) gives

$$A_{m,\mathrm{LMI}}^* = \begin{bmatrix} -0.9259 & 0 & -2.2734 & 0 \\ 0 & -0.9257 & 0 & -2.2733 \end{bmatrix}. \tag{74}$$

Having obtained three different target models, the corresponding model-matching state-feedback gains for each choice are obtained from:

$$F_i = B_{m,i}^{-1}(A_m^* - A_{m,i}), \quad i = 1, \dots, 11, \tag{75}$$

by substituting $A_{m,\mathrm{aver}}^*, A_{m,\max}^*, A_{m,\mathrm{LMI}}^*$ into A_m^*, respectively.

In the second stage of the control design, we define matrices $K_1,\ K_2,\ M$ such that the control law in (68d) is stabilising. The top-down approach is adopted next. Viewing Figure 3, the maximum vertex degree of $\mathcal{G}_{11}$ is $d_{\max} = 4$. For each target system, represented as $(A,\ B)$, we solve LQR problem (A2) for $N_L = 4 + 1$ systems with tuning parameters $(Q_1,\ Q_2,\ R)$. For a particular choice of $(Q_1,\ Q_2,\ R)$, we define $K_1 = -R^{-1}B'P$ and $K_2 = R^{-1}B'\tilde{P}_2$, where $P, \tilde{P}_2$ are obtained from solving ARE (A5), (A6), respectively.

Overall stability of the distributed scheme is guaranteed by an appropriate selection of matrix M. Calculating the eigenvalues of $\mathcal{L}_{G,11}$ and choosing $M = \beta\mathcal{L}_{G,11}$, from Theorem A.1, the closed-loop matrix

$$A_{cl,11} = I_{11} \otimes A + (I_{11} \otimes B)(I_{11} \otimes K_1 + M \otimes K_2), \tag{76}$$

is guaranteed to be Hurwitz for all $\beta > 2.5$.

Simulation results are presented for two different tuning parameters $(Q_1,\ Q_2,\ R)$. The first choice penalises more heavily individual displacements $x_{i,1}, x_{i,2}$ relative to the second by setting $Q_1 = \mathrm{diag}(1, 1, 0, 0)$, $Q_2 = Q_1$, $R = I_2$. Velocity variables are not weighted in this study. In the second choice, relative state information $(x_i - x_j)$ is emphasised in the cost function by selecting $Q_1 = \mathrm{diag}(0.01, 0.01, 0, 0)$, $Q_2 = 10^4 Q_1$ and $R = I_2$. Identical initial conditions are considered for all simulations. Note that the objective here is not to obtain an *optimal* overall design, but rather to illustrate the effects of choosing different tuning parameters.

Mass (m_1) displacements of each oscillator are illustrated in Figures 4, 6, and 8 with LQR performance index tuned to equally

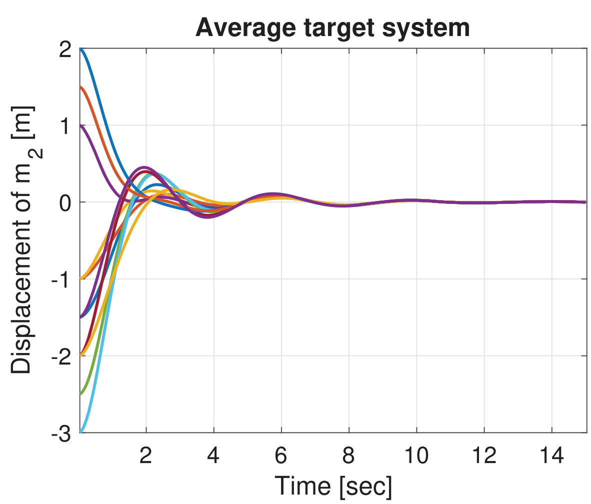

Figure 4. Displacement of m_1 for tuning parameters $Q_1 = Q_2$ and least-squares target selection.

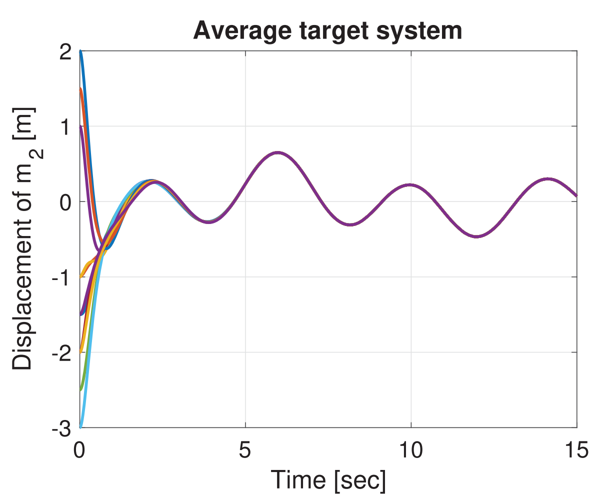

Figure 5. Displacement of m_1 for tuning parameters $Q_2 = 10^4 Q_1$ and least-squares target selection.

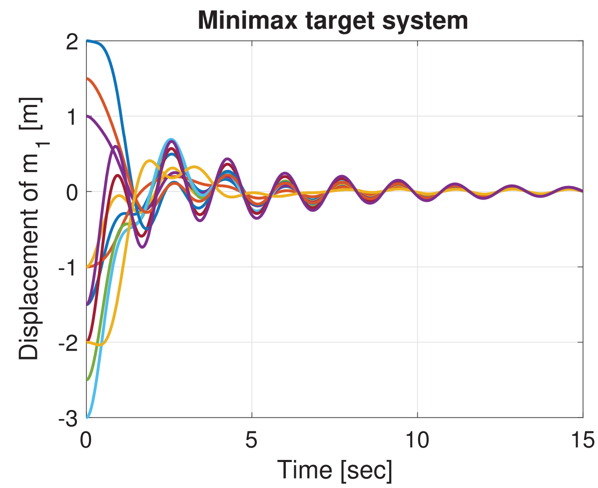

Figure 6. Displacement of m_1 for tuning parameters $Q_1 = Q_2$ and minimax target selection.

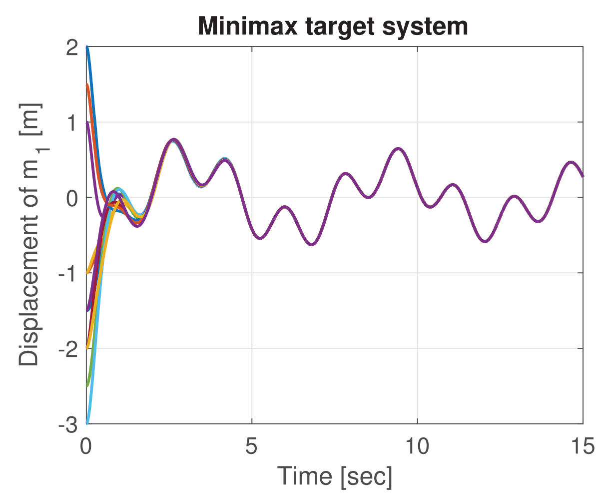

Figure 7. Displacement of m_1 for tuning parameters $Q_2 = 10^4 Q_1$ and minimax target selection.

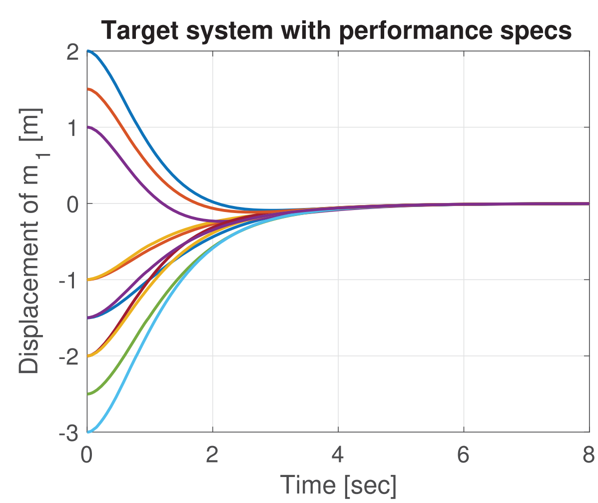

Figure 8. Displacement of m_1 for tuning parameters $Q_1 = Q_2$ and target system with performance specifications.

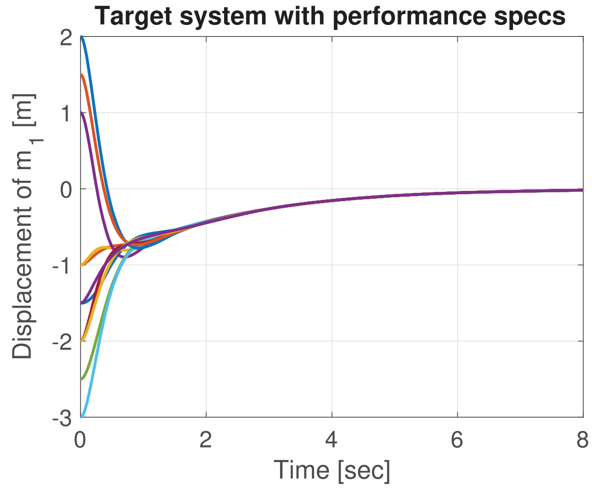

Figure 9. Displacement of m_1 for tuning parameters $Q_2 = 10^4 Q_2$ and target system with performance specifications.

weigh individual and relative state displacements by selecting $Q_1 = Q_2$. As indicated in the figures, the stable operation is irrespective both of the target selection and of the choice of weighting matrices. However, the behaviour of the network is fundamentally altered, as expected, when a target model is selected to satisfy the performance specifications defined earlier. This difference in network's behaviour is illustrated in Figure 8 where systems depict an overdamped response consistent with the specifications, in contrast to the oscillatory behaviour demonstrated when an optimal target model is chosen. This drastic change in network's operation with respect to target selection is amplified when LQR cost function penalises heavily the relative state difference between neighbouring oscillators by selecting a large weighting matrix Q_2. This is evident in Figures 5, 7 and 9. We note that the highly oscillatory behaviour illustrated in these figures stems from the extremely low choice of matrix $Q_1 = 0.0001Q_2$ in the local LQR problem $(A,\ B,\ Q_1,\ R)$ resulting in a closed-loop matrix $A - BR^{-1}B'P$ with some eigenvalues lying near the imaginary axis. This may be rectified, if required, by adjusting appropriately the tuning parameter Q_1. We consider such extreme choices of LQR tuning parameters $(Q_1,\ Q_2,\ R)$ to highlight that the performance of the overall distributed control system is effectively controlled by tuning appropriately the LQR cost function. For example, in all three target selection scenarios, we wish to show that when matrix Q_2 is much larger than Q_1, agents tend first to reach consensus and then converge to the origin. In addition, when a damping specification is imposed on target dynamics both state agreement and target dynamics constraint are satisfied. Thus the simulation results validate our objectives and prove that the model-matching technique is consistent and compatible with a distributed LQR formulation.

7. Conclusion

We have removed the restrictive assumption of identical system dynamics considered in two well-established distributed control methods. Via a two-stage control strategy, we have shown that this technical limitation can be relaxed with more natural requirements. In particular, we assume that agents constituting the network belong to a family of linear systems completely characterised by identical sets of controllability indices, an assumption typically satisfied in many cooperative control applications. The first stage of the method solves model-matching problems and synthesises local state-feedback controllers such that the closed-loop agents are mapped to a target model. In the second stage of the method, the distributed control scheme is designed on the locally regulated systems via either a top-down or a bottom-up method. Any stabilising distributed state-feedback scheme designed on the target dynamics can be adopted in the proposed setting. This feature is indicative of the high flexibility of our approach. The target model can be selected so that perturbations in system dynamics resulting from local state feedback control are minimal. The definition of the target model can then be attained by minimising a certain measure of the joint model-matching control effort. In this respect, closed-loop network performance effectively depends on the selected LQR optimality criterion and minimally on local feedback control. Although this study focuses on multi-agent networks with linear dynamics, it turns out that the proposed model-matching control protocol can readily be extended to the nonlinear case via feedback linearisation control. The latter represents an interesting direction for future research.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendices

Appendix 1. Top-down method

Let the aggregate state-space form of N_L identical linear systems $(A,\ B)$ be given by

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \quad \tilde{x}(0) = \tilde{x}_0, \tag{A1}$$

where $\tilde{A} = I_{N_L} \otimes A$, $\tilde{B} = I_{N_L} \otimes B$, and consider a centralized LQR problem formulated as

$$\underset{\tilde{u}}{\text{minimize}} \quad J(\tilde{u},\ \tilde{x}_0) = \int_0^\infty \left(\tilde{x}'(I_{N_L} \otimes Q_1 + \mathcal{L}_c \otimes Q_2)\tilde{x} + \tilde{u}'(I_{N_L} \otimes R)\tilde{u}\right) \mathrm{d}t \tag{A2a}$$

$$\text{subject to} \quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \tilde{x}(0) = \tilde{x}_0, \tag{A2b}$$

with $Q_1 = Q_1' \geq 0$, $Q_2 = Q_2' \geq 0$, $R = R' > 0$, and $\mathcal{L}_c$ denoting the Laplacian matrix of a complete graph (i.e.a graph with all possible edges). Aggregate state and input vectors are defined as earlier, i.e.$\tilde{x} = \mathrm{Col}(x_1, \ldots, x_{N_L})$, $\tilde{u} = \mathrm{Col}(u_1, \ldots, u_{N_L})$.

Letting now $C_1'C_1 = Q_1$, $C_2'C_2 = Q_2$, and assuming that system (A, B) is controllable and systems (A, C_1), (A, C_2) are observable, problem (A2) has a unique stabilising solution $\tilde{u} = \tilde{K}\tilde{x} = -\tilde{R}^{-1}\tilde{B}'\tilde{P}\tilde{x}$ associated with minimum achievable performance index $J(\tilde{u}, \tilde{x}_0) = \tilde{x}_0'\tilde{P}x_0$, where $\tilde{P}$ is the (unique) stabilising solution to ARE:

$$\tilde{A}'\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}'\tilde{P} + \tilde{Q} = 0. \tag{A3}$$

Due to the special structure of LQR problem (A2), matrix $\tilde{P}$ takes the following form:

$$\tilde{P} = \begin{bmatrix} P - (N_L - 1)\tilde{P}_2 & \tilde{P}_2 & \cdots & \tilde{P}_2 \\ \tilde{P}_2 & P - (N_L - 1)\tilde{P}_2 & & \vdots \\ \vdots & \ddots & \ddots & \tilde{P}_2 \\ \tilde{P}_2 & \cdots & \tilde{P}_2 & P - (N_L - 1)\tilde{P}_2 \end{bmatrix}, \tag{A4}$$

where matrix $P = P' > 0$ is the stabilising solution to ARE:

$$A'P + PA - PBR^{-1}B'P + Q_1 = 0, \tag{A5}$$

associated with a single-node LQR problem with parameters $(A,\ B,\ Q_1,\ R)$. Thus the closed-loop matrix $A_c = A - BR^{-1}B'P$ is Hurwitz. Matrix $\tilde{P}_2$, as appearing in (A4), is a symmetric negative definite matrix associated with

$$A_c'(-N_L\tilde{P}_2) + (-N_L\tilde{P}_2)A_c - (-N_L\tilde{P}_2)BR^{-1}B'(-N_L\tilde{P}_2) + N_LQ_2 = 0, \tag{A6}$$

which can be interpreted as ARE related to an LQR problem with parameters $(A - BR^{-1}B'P,\ B,\ N_LQ_2,\ R)$. Based on (A5) and (A6), we have the following property.

Remark A.1: $\mathcal{A} = A - BR^{-1}B'P + \gamma N_LBR^{-1}B'\tilde{P}_2$ is a Hurwitz matrix for $\gamma = 0$ and $\gamma > 1/2$: $\mathcal{A}$ is clearly Hurwitz for $\gamma = 0$ and $\gamma = 1$ since P and $-N_L\tilde{P}_2$ are stabilising solutions to (A5) and (A6), respectively. The stability condition for $\gamma > 1/2$ stems from the gain-margin property of an LQR problem with parameters $(A - BR^{-1}B'P, B, N_LQ_2, R)$ associated with ARE (A6).

Based on Remark A.1, the following theorem summarises a design procedure of distributed suboptimal LQR-based network control.

Theorem A.1 (Borrelli & Keviczky, 2008)**:** *Let N identical linear agents with aggregate dynamics as shown in* (9a) *form a network with topology described by an undirected graph $\mathcal{G}$ with maximum vertex degree $d_{\max}$. Considering LQR problem* (A2) *with $N_L = d_{\max} + 1$ systems, define matrices P and $\tilde{P}_2$ via* (A5) *and* (A6)*, respectively. Let also $M \in \mathbb{R}^{N\times N}$ be a symmetric matrix* *such that:*

$$\lambda_i(M) > \frac{N_L}{2}, \quad \forall \lambda_i(M) \in S(M)\backslash\{0\}, \tag{A7}$$

and construct a distributed state-feedback gain matrix as

$$\hat{K} = -I_N \otimes R^{-1}B'P + M \otimes R^{-1}B'\tilde{P}_2. \tag{A8}$$

Then, the closed loop system $A_{cl} = I_N \otimes A + (I_N \otimes B)\hat{K}$ is asymptotically stable.

A proof of Theorem A.1 along with several choices of matrix M can be found in Borrelli and Keviczky (2008). Typically, matrix M is chosen to reflect the topology of $\mathcal{G}$ and can be constructed via the adjacency or the Laplacian matrix of $\mathcal{G}$. The result provides a suboptimal solution to problem (5) under the assumption that (A_i, B_i), $i = 1, \ldots, N$ are identical. A complementary approach to distributed LQR-based control is presented next.

Appendix 2. Bottom-up method

Let the state-space forms of N identical linear systems be given by

$$\dot{x}_i = Ax_i + Bu_i, \; x_i(0) = x_{0,i}, \; i = 1, \ldots, N, \tag{A9}$$

where (A, B) is a controllable pair, with $A \in \mathbb{R}^{n\times n}$, $B = [0'_{(n-m)\times m} \; B'_2]' \in \mathbb{R}^{n\times m}$ and $\det(B_2) \neq 0$. Note that the assumed structure of B involves no loss of generality (provided B has full column rank) as can be verified by a change of coordinates.

Assigning each system to a node of graph $\mathcal{G}$, the method proposes a state-feedback control law which, at node level, takes the following form:

$$u_i = Kx_i + \Xi K \sum_{j|j\in\mathcal{N}_i} (x_i - x_j), \quad i = 1, \ldots, N, \tag{A10}$$

where $K \in \mathbb{R}^{m\times n}$ and $\Xi \in \mathbb{R}^{m\times m}$ are to be designed. At network level, the aggregate state-space equations are

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u}, \; \hat{u} = \hat{K}\hat{x}, \tag{A11}$$

where $\hat{K} = I_N \otimes K + \mathcal{L} \otimes \Phi K$ is a distributed state-feedback controller. The problem of finding K and Φ is tackled in two steps, whereby no interaction between agents is initially considered. This allows scaling matrix Ξ in (A10) to be temporarily taken identically zero and controller K to be designed as the optimal state-feedback gain derived from an LQR problem with parameters (A, B, Q_1, R), i.e. $K = -R^{-1}B'P$. Note that $P = P' > 0$ is the solution to ARE:

$$A'P + PA - PBR^{-1}B'P + Q_1 = 0. \tag{A12}$$

Subsequently, letting $\mathcal{L} = V\Lambda V'$ be the spectral decomposition of the Laplacian matrix $\mathcal{L}$, where $V \in \mathbb{R}^{N\times N}$, $V'V = VV' = I_N$, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$, with $\lambda_i \in S(\mathcal{L})$, $i = 1, \ldots, N$, the following state-space transformation is defined: $\hat{\xi} = (V \otimes I_n)\hat{x}$. It can easily be verified that the performance index (5a) can be written as

$$J(\cdot) = \int_0^\infty \left(\hat{\xi}'(I_N \otimes Q_1 + \Lambda \otimes Q_2)\hat{\xi} + \hat{u}'\hat{R}\hat{u}\right) \mathrm{d}t. \tag{A13}$$

Since Λ is diagonal, letting $[\xi'_1, \ldots, \xi'_N]' \triangleq \hat{\xi}$, (A13) reduces to

$$J(\cdot) = \sum_{i=1}^N \int_0^\infty (\xi'_i(Q_1 + \lambda_i Q_2)\xi_i + u'Ru_i) \, \mathrm{d}t. \tag{A14}$$

According to Deshpande et al. (2012), an upper bound of (A14) can be designed as follows. Consider quadratic Lyapunov functions $V_i = \xi'_i P_i \xi_i$, $i = 1, \ldots, N$, where

$$P_i = \begin{bmatrix} P_{i,11} & 0 \\ 0 & \Pi_2 \end{bmatrix} > 0, \quad i = 1, \ldots, N. \tag{A15}$$

Here, $P_{i,11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $\Pi_2 \in \mathbb{R}^{m\times m}$ are assumed to be symmetric positive definite matrices. Note that Π_2 is taken identical for all $i = 1, \ldots, N$. Finally, the second step of the design procedure considers the following optimization problem:

$$\min \sum_{i=1}^N \text{trace}(P_i) \text{ subject to:} \tag{A16a}$$

$$\begin{aligned} &P_i(A + BK + \lambda_i B\Xi K) + (A + BK + \lambda_i B\Xi K)'P_i + (Q_1 + \lambda_i Q_2) \\ &\quad + (K + \lambda_i \Xi K)'R(K + \lambda_i \Xi K) < 0, \quad i = 1, \ldots, N, \end{aligned} \tag{A16b}$$

$$P_i > 0, \quad i = 1, \ldots, N, \tag{A16c}$$

the solution of which yields design matrix Ξ. Note that for random initial state vectors uniformly distributed over the surface of the n-dimensional unit sphere, the expected value of the optimum performance index $J^*(\cdot)$ in (A13) satisfies (Boyd et al., 1994; Deshpande et al., 2012)

$$\mathbb{E}[J^*] \leq \sum_i^N \text{trace}(P_i), \tag{A17}$$

with P_i as given in (A15). Thus, the optimisation problem (A16) represents the minimisation of an upper bound of the global LQR criterion. Detailed description for how to solve optimisation problem (A16) and construct matrix Ξ can be found in Deshpande et al. (2012).

Appendix 3. Proof of Theorem 4.1

We denote by

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{A18}$$

the state-space form of the ith system, with $i = 1, \ldots, N+1$, index $N+1$ referring to the target model. We also consider a change of coordinates $x_{c,i} = P_i x_i$, $i = 1, \ldots, N+1$. Applying

$$u_i = F_{c,i}x_{c,i} + G_i v_i, \tag{A19}$$

for $i = 1, \ldots, N$, the closed-loop state-space form of the i-th system in the new coordinates is written as

$$\dot{x}_{c,i} = (A_{c,i} + B_{c,i}F_{c,i})x_{c,i} + B_{c,i}G_i v_i, \quad x_i = P_i^{-1}x_{c,i}. \tag{A20}$$

We require that

$$A_{c,i} + B_{c,i}F_{c,i} = A_{c,N+1} \quad \text{and} \quad B_{c,i}G_i = B_{c,N+1}, \tag{A21}$$

for $i = 1, \ldots, N$. Since the pairs $(A_{c,i}, B_{c,i})$, $i = 1, \ldots, N+1$, have identical c.i., thereby having identical Brunovsky form denoted as $(\bar{A}_c, \bar{B}_c)$, (A21) leads to

$$F_{c,i} = B_{m,i}^{-1}(A_{m,N+1} - A_{m,i}), \quad G_i = B_{m,i}^{-1}B_{m,N+1}, \tag{A22}$$

where $\det(B_{m,i}) \neq 0$ since, by assumption, $\text{rank}(B_i) = m$, $i = 1, \ldots, N$. From (A21) we also write that

$$P_i(A_i + B_iF_i)P_i^{-1} = P_{N+1}A_{N+1}P_{N+1}^{-1}, \quad P_iB_iG_i = P_{N+1}B_{N+1}, \tag{A23}$$

or

$$A_i + B_iF_i = P_i^{-1}P_{N+1}A_{N+1}P_{N+1}^{-1}P_i, \quad B_iG_i = P_i^{-1}P_{N+1}B_{N+1}, \tag{A24}$$

where

$$F_i = B_{m,i}^{-1}(A_{m,N+1} - A_{m,i})P_i, \tag{A25}$$

for $i = 1, \ldots, N$. Denoting $\Phi_i = P_i^{-1}P_{N+1}$ in (A24) proves (23) while (A22) along with (A25) proves (22).

Appendix 4. Proof of Theorem 4.2

(i) If $(A_i, B_i) \in S(A_o, B_o)$, $i \in \{1, \ldots, N\}$, then

$$A_i = A_o + B_oZ_i \quad \text{and} \quad B_i = B_oG_i^{-1}, \tag{A26}$$

for $Z_i \in \mathbb{R}^{m\times n}$, $G_i \in \mathbb{R}^{m\times m}$, $\det(G_i) \neq 0$. Then, $B_iG_i = B_o$, $\forall i \in \{1, \ldots, N\}$, and hence, $\text{Im}(B_i) = \text{Im}(B_o)$, $\forall i \in \{1, \ldots, N\}$. Let $X = I_n$, $Y_i = -G_iZ_i$.

Then, $\forall(i,j) \in \{1,\dots,N\}^2$,

$$A_iX + B_iY_i - A_jX - B_jY_j = (A_o + B_oZ_i)I_n + B_oG_i^{-1}(-G_iZ_i) \tag{A3}$$

$$- (A_o + B_oZ_j)I_n - B_oG_j^{-1}(-G_jZ_j) = 0, \tag{A28}$$

as required. (ii) Conversely, let $\{(A_i,\ B_i)\}_{i=1}^N$ be given with $\mathrm{Im}(B_i) = \mathcal{X} \subseteq \mathbb{R}^n$, $\dim(\mathcal{X}) = m$. Then, let B_i have a singular value decomposition

$$B_i = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_i \\ 0 \end{bmatrix} V_i', \tag{A29}$$

for $i = 1,\dots,N$, with $U_1 \in \mathbb{R}^{n\times m}$, $\mathrm{Im}(U_1) = \mathrm{Im}(B_i) = \mathcal{X}$, $\mathrm{Im}(U_2) = \mathcal{X}^\perp$, $\det(\Sigma_i) \neq 0$ and $V_i'V_i = V_iV_i' = I_m$. Define $B_o = U_1$, $G_i = V_i\Sigma_i^{-1}$, for $i = 1,\dots,N$. Then,

$$B_iG_i = U_1\Sigma_iV_i'V_i\Sigma_i^{-1} = U_1 = B_o, \tag{A30}$$

which implies that $B_i = B_oG_i^{-1}$, $i = 1,\dots,N$. Further, $\forall(i,j) \in \{1,\dots,N\}^2$:

$$A_iX + B_iY_i - A_jX - B_jY_j = 0, \tag{A31}$$

$$\implies (A_i - A_j)X + B_oG_i^{-1}Y_i - B_oG_j^{-1}Y_j = 0, \tag{A32}$$

$$\implies A_i - A_j = B_o(G_j^{-1}Y_j - G_i^{-1}Y_i)X^{-1}, \tag{A33}$$

$$\implies U_2'(A_i - A_j) = U_2'U_1(G_j^{-1}Y_j - G_i^{-1}Y_i)X^{-1} = 0, \tag{A34}$$

$$\implies A_i - A_j = B_oZ_{ij}, \tag{A35}$$

for some $Z_{ij} \in \mathbb{R}^{m\times n}$. Hence,

$$A_1 - A_2 = B_oZ_{12}, \tag{A36}$$

$$A_2 - A_3 = B_oZ_{23}, \tag{A37}$$

$$\vdots \tag{A38}$$

$$A_{N-2} - A_{N-1} = B_oZ_{N-2,N-1}, \tag{A39}$$

$$A_{N-1} - A_N = B_oZ_{N-1,N}. \tag{A40}$$

Set now $A_o = A_N$, which implies $A_N = A_o + B_o0$. Then,

$$A_{N-1} = A_N + B_oZ_{N-1,N} = A_o + B_oZ_{N-1,N}, \tag{A41}$$

$$A_{N-2} = A_{N-1} + B_oZ_{N-2,N-1} = A_o + B_o(Z_{N-2,N-1} + Z_{N-1,N}), \tag{A42}$$

$$\vdots \tag{A43}$$

$$A_1 = A_2 + B_oZ_{12} = A_o + B_o(Z_{12} + Z_{23} + \cdots + Z_{N-1,N}), \tag{A44}$$

and consequently,

$$(A_i,\ B_i) = S(A_o,\ B_o),\ \forall i = 1,\dots,N. \tag{A45}$$

Appendix 5.Construction of controllability indices of nonlinear systems

Let

$$\dot{x} = \mathbf{f}(x) + \sum_{i=1}^{m} u_i\mathbf{g}_i(x), \quad x(0) = x_0, \tag{A46}$$

be a nonlinear system, where $\mathbf{f}, \mathbf{g}_1,\dots,\mathbf{g}_m$ are smooth vector fields on some neighbourhood $X \subseteq \mathbb{R}^n$ near the origin containing x_0, with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. We define the following; the Lie bracket of $\mathbf{f}$ and $\mathbf{g}_j$ is denoted by $[\mathbf{f},\ \mathbf{g}_j]$ and is the vector field defined by

$$[\mathbf{f},\ \mathbf{g}_j] = \frac{\partial \mathbf{g}_j}{\partial x}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial x}\mathbf{g}_j. \tag{A47}$$

Using this notation, we denote repeated Lie brackets of vector fields $\mathbf{f}$, $\mathbf{g}_1,\dots,\mathbf{g}_m$ as follows:

$$\mathrm{ad}_{\mathbf{f}}^{i+1}\mathbf{g}_j = [\mathbf{f},\ \mathrm{ad}_{\mathbf{f}}^{i}\mathbf{g}_j]. \tag{A48}$$

For instance,

$$\mathrm{ad}_{\mathbf{f}}^{1}\mathbf{g}_j = [\mathbf{f},\ \mathbf{g}_i], \tag{A49}$$

$$\mathrm{ad}_{\mathbf{f}}^{2}\mathbf{g}_j = [\mathbf{f},\ [\mathbf{f},\ \mathbf{g}_j]], \tag{A50}$$

$$\mathrm{ad}_{\mathbf{f}}^{3}\mathbf{g}_j = [\mathbf{f},\ [\mathbf{f},\ [\mathbf{f},\ \mathbf{g}_j]]]. \tag{A51}$$

A set of integers pertaining to vector fields $\mathbf{f}$, $\mathbf{g}_1,\dots,\mathbf{g}_m$ associated with (A46) is constructed via the following procedure.

Procedure A.1: Given system (A46), define the following distributions:

$$C_i = \{\mathrm{ad}_{\mathbf{f}}^{k}\mathbf{g}_j,\ 1 \le j \le m,\ 0 \le k \le i\}, \tag{A52}$$

$$\Delta_i = \mathrm{span}\ C_i, \tag{A53}$$

for $i = 0, 1,\dots,n-1$. Assuming that x_0 is a regular point of the ith distribution Δ_i, $i = 0,\dots,n-1$, compute

$$\Delta_0 = \mathrm{span}\{\mathbf{g}_1,\dots,\mathbf{g}_m\}, \tag{A54}$$

$$\Delta_1 = \mathrm{span}\{\mathbf{g}_1,\dots,\mathbf{g}_m, [\mathbf{f},\ \mathbf{g}_1],\dots,[\mathbf{f},\ \mathbf{g}_1]\}, \tag{A55}$$

$$\vdots$$

$$\Delta_{n-1} = \mathrm{span}\{\mathbf{g}_1,\dots,\mathbf{g}_m, [\mathbf{f},\ \mathbf{g}_1],\dots,[\mathbf{f},\ \mathbf{g}_m],\dots,\mathrm{ad}_{\mathbf{f}}^{n-1}\mathbf{g}_1,\dots, \mathrm{ad}_{\mathbf{f}}^{n-1}\mathbf{g}_m\}. \tag{A56}$$

Define also

$$r_0 = \dim(\Delta_0) = m, \tag{A57a}$$

$$r_i = \dim(\Delta_i) - \dim(\Delta_{i-1}),\ \text{for } i \ge 1. \tag{A57b}$$

Then, the ith integer κ_i, with $i = 1,\dots,m$ is defined as the number of the integers r_i in (A57) that are greater than or equal to i.